

LESSON PLAN 1. Numerical Iteration

Aim: To introduce pupils to the process of iteration and to recognize that the eventual outcome of a particular iteration depends on both the choice of the iteration rule and the choice of the seed.

Teaching objectives:

Content

- Numerical iteration rule
- Seed or initial value of the iteration
- Orbit of the seed under the iteration rule
- Fate of the orbit
- Problem solving and modelling

Communication

- Predicting: “tends to ...”, “will go to...”, “if ... then ...”
- Comparing: “greater than ...”, “between ...”, “less than ...”
- Arguing with partners rules and fates of orbits
- Translating verbal descriptions into algebraic expressions
- Reading summary sentences to other groups

Cognition

- Identifying iteration rules
- Predicting eventual outcomes for various seeds
- Writing summary reports using key vocabulary
- Creating spreadsheets to display data
- Applying iteration to problem solving and modelling
- Arguing and checking answers

Context

- Financial models: appreciation, depreciation, savings accounts and loans
- Healthcare models: cell division, taking medication
- The Tower of Hanoi puzzle

Outcomes: At the end of the lesson, pupils will be able to

- Know each step of an iterative process
- Identify patterns in different contexts
- Translate verbal expressions to algebraic notation
- Use spreadsheets to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in scientific English language
- Work in pairs effectively
- Recognise the importance of arguing in a team
- Know how to behave in a discussion

Key words: iteration rule, seed, orbit, fate, tends to infinity.

Prerequisites: Pupils need to be familiar with whole numbers, positives and negatives, decimal numbers and percents.

Curriculum connections:

- Computation with integers, decimals and percents in an investigative setting rather than as a drill.
- Description of algorithms, use of variables and introduction to symbolic representation.
- Practise with problem solving and recognizing patterns.
- Use of calculator or computer when needed.

Timing: Six 55-minute class periods.

Tasks planned:

First session

- A power-point introducing the key words and the main ideas of an iterative process is shown. While showing this power-point, the teacher asks questions to the pupils to be sure that they are entering into the basic terminology.
- Pupils work through exercise 1-*Doubling rule* individually.
- Plenary: correction of exercise 1 and discussion about it.
- After having pupils working on exercises 2-*Halving rule*, 3-*Adding five* and 4-*Sign changing*, focus on arrow notation, reading rules aloud.
- Homework: Practise with 6-*Some more iteration rules*. Ask to prepare a dictation of 4 iteration rules, similar to ones from exercise 5-*Arrow notation*, to be done the next session.

Second session

- Each pupil reads aloud their 4 iteration rules and the others write down the arrow notation expression.
- Pupils work in pairs to finish the worksheet 6-*Some more iteration rules* and go through exercise 7-*Squaring rule*, 8-*Square root rule* and 9-*Cubing rule*. Meanwhile teacher revises homework.
- Plenary: summary discussion.
- Introducing exercise 10-*Spreadsheets* at the computer room.
- Homework: Exploring exercises 11-*Counting edges* and 12-*Counting faces*.

Third session

- Pupils contrast in pairs the ending of homework exercises and the teacher circulates among them to be sure they are headed in the right direction.
- Starting with social life exercises 13-*Appreciation* and 14-*Depreciation*, the teacher must go over their results to make sure pupils understand percentages, “original amount” and “new amount”.
- If there is not enough time in class to finish healthcare exercises 15-*Taking medication* and 16-*Daily doses*, they can be left as homework.

Fourth session

- Pupils introduce themselves gradually to compound interest through exercises 17-*Savings account*, 18-*Savings plan*, 19-*Doubling your investment* and 20-*Loans*. Work at the computer is needed.
- Plenary: summary discussion.
- Homework: Ask pupils to finish the remaining exercises.

Fifth session

- Give time to understand 21-*The Tower of Hanoi* game and to provide pieces to play hands on.
- Before the ending of the class point out the website to run simulations of the game.
- Homework: conclusions about the solution.

Sixth session

- Search for information on the Web about The Tower of Hanoi history and answer the questions.
- Revision of conclusions in cooperative groups.
- Plenary: teacher helps pupils to summarise the iteration underlying into The Tower of Hanoi problem.

Resources

- For the power-point: a computer, a screen and a projector.
- For class activities and homework: the pupils' worksheets and access to computers.
- For The Tower of Hanoi problem: cardboard, compasses, scissors, coloured pencils ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Iteration* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Power-point presentation

This Power-point presentation starts automatically. From one slide to the next one you have to click "Enter", so you have time for explanations and comments.

First slide: ITERATION

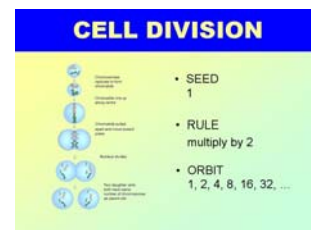
Unit's title explanation.



Second slide: CELL DIVISION

Cell division is a simple example of a repeating process. This process can be **modelled** using the key words "iteration rule", "seed", "orbit".

Comment on phrases using these words like "class rules", "plant a seed", "tennis: seed player", "planet's orbit" and so on.

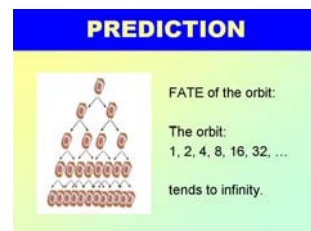


Third slide: PREDICTION

Modelling allows us to make predictions what would happen if we continued iterating.

Doubling the previous result, the process gets very large values but we **don't** calculate "a final value".

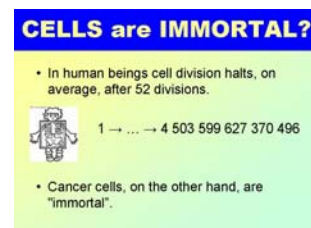
Relate to sentences like "I met my girlfriend at the disco. She is now my wife. It was fate."



Fourth slide: CELLS ARE IMMORTAL?

Bear in mind that we are working in cell division **context**!

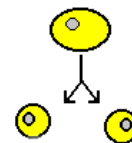
Ask pupils to check the big number showed on the slide. It's an opportunity to verify that big numbers are better known in scientific notation.



If the power-point is not available you can use the cell division example to introduce the basic concepts in an interactive way.

Answers to exercises1-Doubling rule

This is the iteration rule of the power-point presentation.
 Point out that the initial value can be changed in other applications.
 Omit negative numbers if your pupils have not yet studied this topic.
 Encourage mental calculation.



1. Using the iteration rule **multiply by 2**, write the first ten numbers in the orbits for the following seeds:

Seed = 1 Orbit: **1, 2, 4, 8, 16, 32, 64, 128, 256, 512**

Seed = 1.5 Orbit: **1.5, 3, 6, 12, 24, 48, 96, 192, 384, 768**

Seed = 0 Orbit: **0, 0, 0, 0, 0, 0, 0, 0, 0, 0**

Seed = -1 Orbit: **-1, -2, -4, -8, -16, -32, -64, -128, -256, -512**

2. Explain the **eventual outcomes** for these various seeds. The boxes below will help you.

Pupils must recognize that, starting by seeds 1 or 1.5, the numbers generated get successively larger and we say they tend to infinity, which is not a final value.

- **The orbit of 1 tends to positive infinity**
- **The orbit of 1.5 tends to positive infinity**
- **The orbit of 0 stays fixed**
- **The orbit of -1 tends to negative infinity**

The orbit of tends to positive infinity

The orbit of tends to negative infinity

The orbit of tends to zero

The orbit of stays fixed

3. **Predict** the eventual outcomes using the seeds 6 and -6. Use the boxes above and the future ...**will**

- **The orbit of 6 will tend to positive infinity**
- **The orbit of -6 will tend to negative infinity**

4. Did all the seeds produce the same behaviour? Explain.

- **If the seed is positive, the orbit will tend to positive infinity**
- **If the seed is negative, the orbit will tend to negative infinity**
- **If the seed is zero, the orbit stays fixed**

2-Halving rule

1. Using the iteration rule **divide by 2**, write the first eight numbers in the orbits for the following seeds:

Seed = 10 Orbit: **10, 5, 2.5, 1.25, 0.625, 0.3125, 0.15625, 0.078125**

Seed = 0 Orbit: **0, 0, 0, 0, 0, 0, 0, 0**

Seed = -4 Orbit: **-4, -2, -1, -0.5, -0.25, -0.125, -0.0625, -0.03125**

2. What is the **eventual outcome** of the iteration in each case?

- **The orbit of 10 tends to zero**
- **The orbit of 0 stays fixed**
- **The orbit of -4 tends to zero**

3. Did all the seeds produce the same behaviour? Explain.

Cutting in half the numbers generated get successively closer and closer to 0, but will never become zero.

The orbit of any seed tends to zero, except the orbit of 0 that stays fixed.

3-Adding five

1. Using the iteration rule **add 5**, write the first nine numbers in the orbit for the following seeds:



Seed = 40 Orbit: **40, 45, 50, 55, 60, 65, 70, 75, 80**

Seed = 0 Orbit: **0, 5, 10, 15, 20, 25, 30, 35, 40**

Seed = -400 Orbit: **-400, -395, -390, -385, -380, -375, -370, -365, -360**

2. Comment on the **fates** of the orbits.

The orbit of any seed tends to positive infinity.

All the rules “adding a positive number” produce the same behaviour.

4-Sign changing

1. Using the iteration rule **sign changing**, write the first ten numbers in the orbits for the following seeds:

Seed = 3 Orbit: **3, -3, 3, -3, 3, -3, 3, -3, 3, -3**

Seed = -4 Orbit: **-4, 4, -4, 4, -4, 4, -4, 4, -4, 4**

Seed = 0 Orbit: **0, 0, 0, 0, 0, 0, 0, 0, 0, 0**

2. We call the orbits of the seeds 3 and -4 **cycles of period 2**. Why?

The first orbit hops back and forth between the two values 3 and -3.

The second orbit has de same behaviour.

3. What can you say about the orbit of the seed 0?

The orbit of 0 stays fixed.

5-Arrow notation

Write each rule using arrow notation and **check it** with your partner:

Pupils must learn to say aloud this rules to their partners and those have to follow them. Omit the arrow notation if your pupils don't feel confident using "x".

Rule	Arrow notation
Add three	$x \rightarrow x + 3$
Divide by four, then add two	$x \rightarrow \frac{x}{4} + 2$
Multiply by minus five	$x \rightarrow -5x$
Take the opposite, then add one	$x \rightarrow -x + 1$
Subtract seven point five	$x \rightarrow x - 7.5$
Subtract six, then multiply by four	$x \rightarrow (x - 6) \cdot 4$
Subtract six, then divide by two	$x \rightarrow (x - 6) / 2$
Multiply by three point five	$x \rightarrow 3.5x$
Add two thirds	$x \rightarrow x + 2/3$

6-Some other rules

This exercise is just to reinforce the process.

For each of the following iteration rules, write the first ten numbers in the seed's orbits, and describe the fate of the orbit.

1. Rule: $x \rightarrow \frac{x}{5}$ Seed = 5000

Orbit: **5000, 1000, 200, 40, 8, 1.6, 0.32, 0.064, 0.0128, 0.00256**

Fate: **The orbit of 5000, under the rule “divide by 5”, tends to zero**

2. Rule: $x \rightarrow 5x$ Seed = 0.5

Orbit: **0.5, 2.5, 12.5, 62.5, 312.5, 1 562.5, 7 812.5, 39 062.5, 195 312.5, 976 562.5**

Fate: **The orbit of 0.5 under the rule “multiply by 5” tends to positive infinity**

3. Rule: $x \rightarrow 3x + 6$ Seed = - 3

Orbit: **- 3, - 3, - 3, - 3, - 3, - 3, - 3, - 3, - 3, - 3**

Fate: **The orbit of - 3 under the rule “multiply by 3, then add 6” stays fixed**

4. Rule: $x \rightarrow -x + 1$ Seed = 1

Orbit: **1, 0, 1, 0, 1, 0, 1, 0, 1, 0**

Fate: **The orbit of 1, under the rule “take the opposite, then add 1”, is a cycle of period 2**

Work in pairs:

Pupils can go back to the rules above.

1. Find one iteration rule that the orbit of all seeds, except 0, is approaching 0.

One rule is “divide by 3” or $x \rightarrow x / 3$

2. Find one iteration rule that works for the orbit: $1 \rightarrow 3 \rightarrow 7 \rightarrow 15 \rightarrow 31 \rightarrow 63 \rightarrow 127 \rightarrow 255 \rightarrow \dots$

One rule is “multiply by 2, then add 1” or $x \rightarrow 2x + 1$

3. Find one iteration rule that works for the orbit: $1 \rightarrow 2 \rightarrow 5 \rightarrow 14 \rightarrow 41 \rightarrow 122 \rightarrow 365 \rightarrow \dots$

One rule is “multiply by 3, then subtract 1” or $x \rightarrow 3x - 1$

7-Squaring rule

Big numbers and small numbers are better known in scientific notation.
Depending on the level of your pupils you may ask for calculations to 2 decimal places.

1. Using the **squaring** iteration rule, $x \rightarrow x^2$, write the first six numbers in the orbits for the following seeds:

Seed = 1	Orbit: 1, 1, 1, 1, 1, 1
Seed = - 1	Orbit: - 1, 1, 1, 1, 1, 1
Seed = 0	Orbit: 0, 0, 0, 0, 0, 0
Seed = 2	Orbit: 2, 4, 16, 256, 65 536, 4 294 967 296
Seed = - 2	Orbit: - 2, 4, 16, 256, 65 536, 4 294 967 296
Seed = 3	Orbit: 3, 9, 81, 6 561, 43 046 721, $1.853 \cdot 10^{15}$
Seed =10	Orbit: 10, 100, 10 000, 100 000 000, 10 000 000 000 000, 10^{32}
Seed = - 0.1	Orbit: - 0.1, 0.01, 0.000 1, 0.000 000 01, 0. 000 000 000 000 000 1, 10^{-32}
Seed = - 0.5	Orbit: - 0.5, 0.25, 0.062 5, 0.003 906 25, $1.5259 \cdot 10^{-5}$, $2.3283 \cdot 10^{-10}$
Seed = 0.5	Orbit: 0.5, 0.25, 0.062 5, 0.003 906 25, $1.525 9 \cdot 10^{-5}$, $2.328 3 \cdot 10^{-10}$
Seed = 0.2	Orbit: 0.2, 0.04, 0.001 6, 0.000 002 56, $6.5536 \cdot 10^{-12}$, $4.295 \cdot 10^{-23}$

2. **Discuss** with your partner what the fate of these orbits is and **classify** the seeds above according to the same behaviour.

Seeds 0 and 1: The orbit stays fixed.

Seed - 1: The orbit itself is not fixed, but after the first number it becomes fixed.

Seeds 2, - 2, 3 and 10: The orbit tends to positive infinity.

Seeds - 0.1, - 0.5, 0.5 and 0.2: The orbit tends to zero.

3. **Predict** what will happen to the orbit of **any** seed under this iteration rule. The word bank below will help you.

If the seed is greater than 1 or less than - 1 the orbit will tend to positive infinity.

If the seed is between - 1 and 1, except 0, the orbit will tend to zero.

If the seed is equal to 0 or 1, the orbit will be fixed.

If the seed is equal to -1 the orbit itself is not fixed, but after the first number it becomes fixed.

Greater than	between	equal to	less than
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This behaviour of the squaring rule should be well understood and even memorised.

8-Square root rule

- Use a calculator to write the first six numbers in the orbits (rounded to two decimal places) for the following seeds under the **square root** iteration rule, $x \rightarrow \sqrt{x}$:



Seed = 1	Orbit:	1, 1, 1, 1, 1, 1
Seed = 0	Orbit:	0, 0, 0, 0, 0, 0
Seed = 16	Orbit:	16, 4, 2, 1.41, 1.19, 1.09
Seed = 10 000	Orbit:	10 000, 100, 10, 3.16, 1.78, 1.33
Seed = 0.1	Orbit:	0.1, 0.32, 0.56, 0.75, 0.87, 0.93
Seed = 0.9	Orbit:	0.9, 0.95, 0.97, 0.99, 0.99, 1

Pay attention that we iterate the result given by the calculator, not the rounded number.

- What do you think will happen to **any positive** seed under this iteration rule?

The orbit of any positive seed will tend to 1!

9-Cubing rule



- Using the **cubing** iteration rule, $x \rightarrow x^3$, write the first five numbers in the orbits for the following seeds:

Seed = 1	Orbit:	1, 1, 1, 1, 1
Seed = -1	Orbit:	-1, -1, -1, -1, -1
Seed = 0	Orbit:	0, 0, 0, 0, 0
Seed = 2	Orbit:	2, 8, 512, 134 217 728, $2.41785 \cdot 10^{24}$
Seed = -2	Orbit:	-2, -8, -512, -134 217 728, $-2.41785 \cdot 10^{24}$
Seed = 0.1	Orbit:	0.001, 0.000 000 001, $1 \cdot 10^{-27}$, $1 \cdot 10^{-81}$

Seed = -0.1 Orbit: $-0.001, -0.000\ 000\ 001, -1 \cdot 10^{-27}, -1 \cdot 10^{-81}$

Seed = -0.5 Orbit: $-0.5, -0.125, -0.001953125, -7.45058 \cdot 10^{-9}, -4.1359 \cdot 10^{-25}$

Seed = 0.5 Orbit: $0.5, 0.125, 0.001953125, 7.45058 \cdot 10^{-9}, 4.1359 \cdot 10^{-25}$

2. **Predict** what will happen to the orbit of **any** seed under this iteration rule.

Compare with 7-Squaring rule.

Seeds 1, -1 and 0: The orbit stays fixed.

Seed 2: The orbit tends to positive infinity.

Seed -2: The orbit tends to negative infinity.

Seeds 0.1, -0.1, -0.5 and 0.5: The orbit tends to zero.

Prediction:

If the seed is greater than 1 the orbit will tend to positive infinity.

If the seed is less than -1 the orbit will tend to negative infinity.

If the seed is between -1 and 1, the orbit will tend to zero.

If the seed is equal to 1, -1 and 0, the orbit will be fixed.

10-Spreadsheets



Numerical iteration lends itself quite well to introducing spreadsheets. Pupils will quickly realise that spreadsheets are very useful in accounting tasks.

Use a computer to check the previous exercises:

1. - Open a spreadsheet and enter the seed 1 into the first cell A1.
- Enter the doubling iteration rule, given by the formula =A1*2, into the second cell A2.

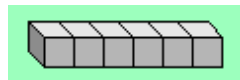
	A	B	C	D	E	F
1	1					
2	=A1*2					
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						

- Dragging this cell down the column you create the orbit of the seed 1 under the doubling rule.
- Enter the seed 1.5 into the first cell of the second column B1.
- Drag the cell A2 to the cell B2.
- Dragging B2 down you create in the second column B the orbit of the seed 1.5.
- Repeat this process entering the seeds 0, - 1, 6, - 6.

Observe the fates of these six orbits and **check** your previous answer.

2. Open a new spreadsheet and proceed as above to check the exercise 2-Halving rule using the formula =A1/2.
3. Open a new spreadsheet and check the exercise 3-Adding five using the formula =A1+5.
4. Open a new spreadsheet and check the exercise 7-Squaring rule using the formula =A1^2.
5. Open a new spreadsheet and check the exercise 8-Square root rule using the formula =SQRT(A1).
6. Open a new spreadsheet and check the exercise 9-cubing rule using the formula =A1^3.

11-Counting edges

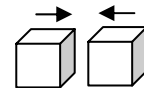


1. How many **edges** does a **cube** have?



12

2. If you glue **two** cubes as at the picture, how many edges do you **add**?

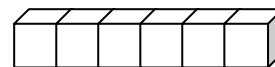


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3. Glue a third cube. How many edges does the shape have?

$$12 + 8 + 8 = \underline{28}$$

4. How many edges does the figure at the picture have?



$$12 + 8 \cdot 5 = \underline{52}$$

5. Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Add 8** or $x \rightarrow x + 8$

Seed: **12**

Orbit: **12, 20, 28, 36, 44, 52**

6. Calculate **in pairs** how many edges a similar structure made up of 100 cubes have. Explain your strategy.

$$12 + 8 \cdot 99 = \underline{804}$$

12-Counting faces

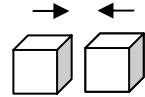
This time focus on counting the **square faces** on the **outside** of each structure.

1. Write down by yourself the six questions of the exercise as above in a blank sheet.
2. Answer them as before.

1. How many faces does a cube have?

6

2. If you glue two cubes as at the picture, how many faces do you add?

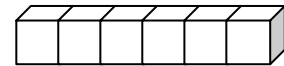


4

3. Glue a third cube. How many faces does the shape have?

$$6 + 4 + 4 = \underline{14}$$

4. How many faces does the figure at the picture have?



$$6 + 4 \cdot 5 = \underline{26}$$

5. Notice that you have a numerical iteration going on. Determine with your partner:

Iteration rule: Add 4 or $x \rightarrow x + 4$

Seed: 6

Orbit: 6, 10, 14, 18, 22, 26

6. Calculate in pairs how many faces a similar structure made up of 100 cubes have. Explain your strategy.

$$6 + 4 \cdot 99 = \underline{402}$$

Some goods rise or appreciate their value with time. Ask pupils for examples. Others drop or depreciate. Exercises 13 and 14 below involve constant appreciation or depreciation, so they may be related to iteration.

The key is to understand the relationship between the original amount = 100% and the new amount.

13-Appreciation

1. A house valued at six hundred and fifty thousand euros **appreciates** for three years by 7% every year. Calculate the value of the house after those three years.



Original amount = 100%

New amount = 107%

First year appreciation: 107 % of 650 000 = 1.07 times 650 000 = 695 500 euros

Second year appreciation: 107 % of 695 500 = **1.07 times 695 500 = 744 185**

Third year appreciation: 107 % of **744 185 = 1.07 times 744 185 = 796 277.95 ≈ 796 278 euros**

Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Multiply by 1.07** or $x \rightarrow 1.07 x$

Seed: **650 000**

Orbit: **650 000, 695 500, 744 185, 796 277.95**

2. A Miquel Barcelo painting, bought for 6 000 euros, appreciated by 150% per year during three years. What was the painting worth after three years?

Original amount = 100%

New amount = 250%



First year appreciation: 250 % of 6 000 = 2.5 times 6 000 = 15 000

Second year appreciation: 250 % of 15 000 = 2.5 times 15 000 = 37 500

Third year appreciation: 250 % of 37 500 = 2.5 times 37 500 = 93 750 euros

Homework: Find information about Miquel Barceló in the Internet.

3. The pressure in a boiler is 180 pascals. A faulty valve causes the pressure to **rise** in the boiler by 12% every hour. The situation will become dangerous when the pressure reaches 300 pascals. If it continues to rise this way, during which hour will the boiler's pressure reach danger level?

Original amount = 100%

New amount = 112%



First hour rise: 112 % of 180 = 1.12 times 180 = 201.6

Second hour rise: 112 % of 201.6 = 1.12 times 201.6 = 225.792

Third hour rise: 112 % of 225.792 = 1.12 times 225.792 = 252.88704

Fourth hour rise: 112 % of 252.88704 = 1.12 times 252.88704 = 283.2334848

Fifth hour rise : 112 % of 283.2334848 = 1.12 times 283.2334848 ≈ 317,221503

317 > 300

The boiler's pressure will reach danger level during the fifth hour.

14-Depreciation

1. A car is originally worth 12 900 euros. It **depreciates** over three years by 7% every year. Calculate its worth after those three years.



Original amount = 100%

New amount = 93%

First year depreciation: 93 % of 12 900 = 0.93 times 12 900 = **11 997**

Second year depreciation: 93 % of **11 997** = **0.93 times 11 997 = 11 157.21**

Third year depreciation: **93 % of 11 157.21 = 0.93 times 11 157.21 = 9 649.87093 ≈ 9 650 euros**

Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Multiply by 0.93** or $x \rightarrow 0.93 x$

Seed: **12 900**

Orbit: **12 900, 11 997, 11 157, 9 650 rounded to the nearest whole number**

2. Peter bought his first motorbike at the start of 2005 for 9 400 euros. It **depreciated** in value by 15% every year he owned it. What was the bike worth by the end of 2007?



Original amount = 100%

New amount = 85%

First year depreciation: 85 % of 9 400 = 0.85 times 9 400 = 6 791.5

Second year depreciation: 85 % of 6 791.5 = 0.85 times 6 791.5 = 5 772.775

Third year depreciation: 85 % of 5 772.775 = 0.85 times 5 772.775 ≈ **4 907 euros**

3. A hot air balloon was drifting along at a height of 5 000 m when it developed a leak. The balloon **dropped** by 8% every minute after that. What was the balloon's height after three minutes?



Original height = 100%

New height = 92%

First minute drop: 92 % of 5 000 = 0.92 times 5 000 = 4 600

Second minute drop: 92 % of 4 600 = 0.92 times 4 600 = 4 232

Third minute drop: 92 % of 4 232 = 0.92 times 4 232 = **3 893.44 metres**

In the next two exercises we change the context, but we keep the same ideas.

15-Taking medication

After you swallow a 100-units capsule, the medication is slowly absorbed by your body. Suppose 25% of the amount in your system is **eliminated** every hour.

- How much of the medication **remains** in your system after 5 hours?



Original amount = 100%

New amount = 75%

After 1 hour: 75 % of 100 = 0.75 times 100 = 75 units

After 2 hours: 75 % of 75 = 0.75 times 75 = **56.25**

After 3 hours: **75 % of 56.25 = 0.75 times 56.25 = 42.187 5**

After 4 hours: **75 % of 42.187 5 = 0.75 times 42.1875 = 31.640 625**

After 5 hours: **75 % of 31.640 625 = 0.75 times 31.640 625 = 23.730 468 8 ≈ 23.73 units**

- Notice that you have a **numerical iteration** that describes the amount of medication in your body. Determine **with your partner**:

Iteration rule: **Multiply by 0.75** or $x \rightarrow 0.75 x$

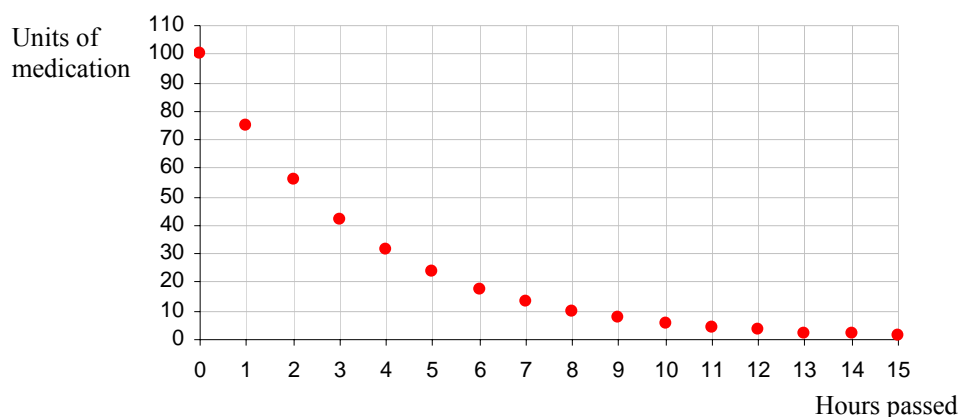
Seed: **75**

Orbit: **75, 56.25, 42.19, 31.64, 23.73 rounded to two decimal places**

- Use a spreadsheet or calculator to compute the amount of medication in your body the next 10 hours.

17.80, 13.35, 10.01, 7.51, 5.63, 4.22, 3.17, 2.38, 1.78, 1.33 rounded to 2 d.p.

- Sketch a plot of these 15 points.



- What is the fate of this orbit?

The orbit tends to zero.

16-Daily doses

Your doctor prescribes you the following medication: “Take an initial dose of 75 units and 25 units every day after”. It is known that your body eliminates 20% of the medication every day.

- How much is the amount of medication in your system after 4 days?



Original amount = 100%

New amount = 80%

After 1 day: 80 % of 75 = 0.8 times 75 = 60 units
60 + 25 = 85 units

After 2 days: 80 % of 85 = **0.8 times 85 = 68**
68 + 25 = 93

After 3 days: **80 % of 93 = 0.8 times 93 = 74.4**
74.4 + 25 = 99.4

After 4 days: **80 % of 99.4 = 0.8 times 99.4 = 79.52**
79.52 + 25 = 104.52 units

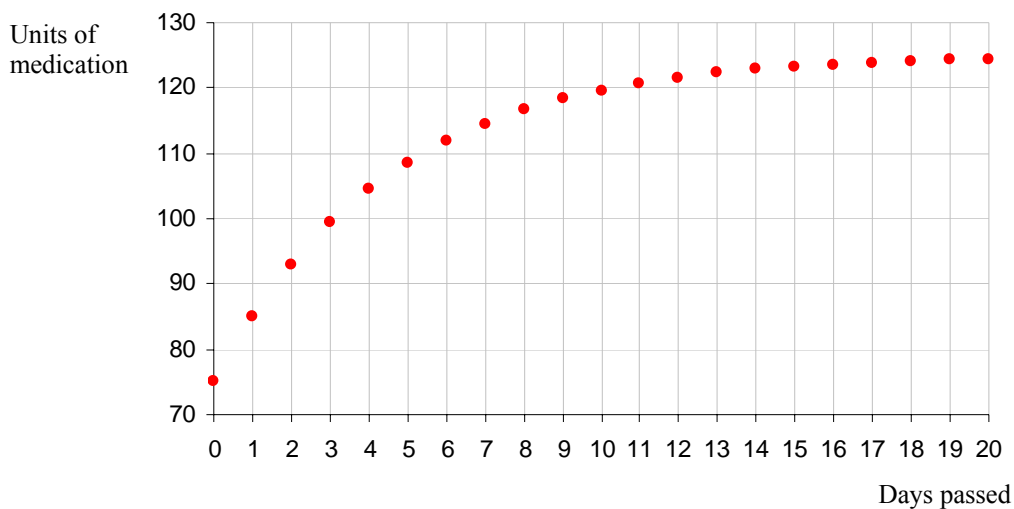
- Notice that you have a **numerical iteration** that describes the amount of medication in your body. Determine **with your partner**:

Iteration rule: **Multiply by 0.8, then add 25** or $x \rightarrow 0.8x + 25$
Seed: **75**
Orbit: **75, 85, 93, 99.4, 104.52**

- If the **toxic level** in your body is 130 units, is it **safe** for you to continue taking this medication? Use a spreadsheet or calculator and **argue** with your partner.

The orbit appears to stabilize around 125 units, which is less than 130, so you are safe.

- Sketch a plot of yours answers.



Financial models are other real world applications. The iterative method is an intuitive way of describing interest compounded. Depending on your teacher style and the level of your pupils, you don't need at the moment to distinguish between simple interest and compound interest.

Again the key is to understand the relationship between the original amount = 100% and the new amount. Then proceed as the exercises above.

All money results must be given rounded to 2 decimal places.

Point out that "p.a." comes from Latin "per annum".

17-Savings account

Special
Savings
Rate
4.2 % p.a.

1. A bank is paying 4.2% **interest** per year on your savings account, and the interest is paid once per year at the end of the year.
You have 800 euros to deposit in the account.

You want to know how much money you will have at the end of **each** year over a five-year period.

Original amount = 100%

New amount = 104.2%

First year balance: 104.2 % of 800 = 1.042 times 800 = 833.6 euros

Second year balance: 104.2 % of 833.6 = 1.042 times 833.6 = 868.61 euros

Third year balance: 104.2 % of 868.61 = 1.042 times 868.61 = 905.09 euros

Fourth year balance: 104.2 % of 905.09 = 1.042 times 905.09 = 943.11 euros

Fifth year balance: 104.2 % of 943.11 = 1.042 times 943.11 = 982.72 euros

Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Multiply by 1.042 or $x \rightarrow 1.042 x$**

Seed: **800**

Orbit: **800, 833.6, 868.61, 905.09, 943.11, 982.72**

Use a spreadsheet or calculator to find the balance in your account after ten years.

1207.17 euros

2. Joan was told that if she left her savings of 2400 euros in the Scotia Bank for 5 years they would give her a special annual **rate** of 5.4%. How much would her 2400 euros be worth at the end of the 5 year period?

Original amount = 100%

New amount = 105.4%

First year balance: 105.4 % of 2400 = 1.054 times 2400 = 2529.6 euros

Second year balance: 105.4 % of 2529.6 = 1.054 times 2529.6 = 2635.84 euros

Third year balance: 105.4 % of 2635.84 = 1.054 times 2635.84 = 2746.55 euros

Fourth year balance: 105.4 % of 2746.55 = 1.054 times 2746.55 = 2861.90 euros

Fifth year balance: 105.4 % of 2861.90 = 1.054 times 2861.90 = 2982.10 euros

18-Savings plan

Suppose that the interest rate in a savings account is 5% annually and that your initial deposit is 4500 euros. If you add 500 euros to this account at the end of each year, what is your balance at the end of the first 6 years?



Original amount = 100%

New amount = 105%

First year balance: 105 % of 4500 = 1.05 times 4500 = 4725

$$\begin{array}{r} + 500 \\ \hline 5225 \end{array}$$

Second year balance: 105 % of 5225 = **1.05 times 5225 = 5486.25**

$$\begin{array}{r} + 500 \\ \hline 5986.25 \end{array}$$

Third year balance: **105 % of 5986.25 = 1.05 times 5986.25 = 6285.56**

$$\begin{array}{r} + 500 \\ \hline 6785.56 \end{array}$$

Fourth year balance: **105 % of 6785.56 = 1.05 times 6785.56 = 7124.84**

$$\begin{array}{r} + 500 \\ \hline 7624.84 \end{array}$$

Fifth year balance: **105 % of 7624.84 = 1.05 times 7624.84 = 8006.08**

$$\begin{array}{r} + 500 \\ \hline 8506.08 \end{array}$$

Sixth year balance: **105 % of 8506.08 = 1.05 times 8506.08 = 8931.39**

$$\begin{array}{r} + 500 \\ \hline 9431.39 \end{array}$$
 euros

Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Multiply by 1.05, then add 500** or $x \rightarrow 1.05x + 500$

Seed: **4500**

Orbit: **4500, 5225, 5986.25, 6785.56, 7624.84, 8506.08, 9431.39**

Use a spreadsheet or calculator to find the balance in your account after ten years.

13 618.97 euros

19-Doubling your investment

1. An internet bank offers a tremendous interest rate of 9% per year on savings. If you invest your savings of 6000 euros in the internet bank:

- a) How much will be your savings worth after 1 year?

Original amount = 100% **New amount = 109%**

109 % of 6000 = 1.09 times 6000 = 6540 euros

- b) How much will be your savings worth after 2 years?

109 % of 6540 = 1.09 times 6540 = 7128.6 euros

- c) How much will be your savings worth after 3 years?

109 % of 7128.6 = 1.09 times 7128.6 = 7770.17 euros

- d) What iteration rule computes account balances using this interest rate?

Multiply by 1.09 or $x \rightarrow 1.09 x$

- e) How many years does it take for you to have more than 12 000 euros in the bank?

Orbit: 6000, 6540, 7128.6, 7770.17, 8469.49, 9231.74, 10062.60, 10968.23, 11955.38, 13031.36

Nine years

- f) In how many years will you have 15 000 euros in the bank?

Eleven years

2. The Scotia Bank offers its costumers an interest rate of 4.5% p.a. on their savings. Ted deposits 1200 euros in a new Scotia bank account. How many years will it take to **double** his investment?

Original amount = 100% **New amount = 104.5%**

Iteration rule: Multiply by 1.045

Seed: 1200

Orbit: 1200, 1254, 2426.84

Sixteen years

20-Loans

Notice that if you borrow money from a bank, the initial balance in your account is negative because you owe the bank money.

Mr. Brown **borrow**s 12 000 euros **from** a bank to buy a car. The interest rate is 1% per month and he makes monthly payments of 300 euros.

- List his loan balances after the first 5 months.

Original amount = 100%

New amount = 101%

Initial balance: -12 000

After 1 month: 101 % of -12 000 = 1.01 times -12 000 = - 12 120

$$- 12 120 + 300 = - 11 820 \text{ euros}$$

After 2 months: 101 % of -11 820 = 1.01 times -11 820 = - **11 938.2**

$$- 11 938.2 + 300 = \underline{- 11 638.2} \text{ euros}$$

After 3 months: **101 % of -11 638.2 = 1.01 times -11 638.2 = - 12 754.58**

$$- 12 754.58 + 300 = \underline{- 11 454.58} \text{ euros}$$

After 4 months: **101 % of -11 454.58 = 1.01 times -11 454.58 = - 11 569.13**

$$- 11 569.13 + 300 = \underline{- 11 269.13} \text{ euros}$$

After 5 months: **101 % of - 11 269.13 = 1.01 times -11 269.13 = - 11 .381.82**

$$- 11 381.82 + 300 = \underline{- 11 081.82} \text{ euros}$$

- Notice that you have a **numerical iteration** going on. Determine **with your partner**:

Iteration rule: **Multiply by 1.01, then add 300** or $x \rightarrow 1.01 x + 300$

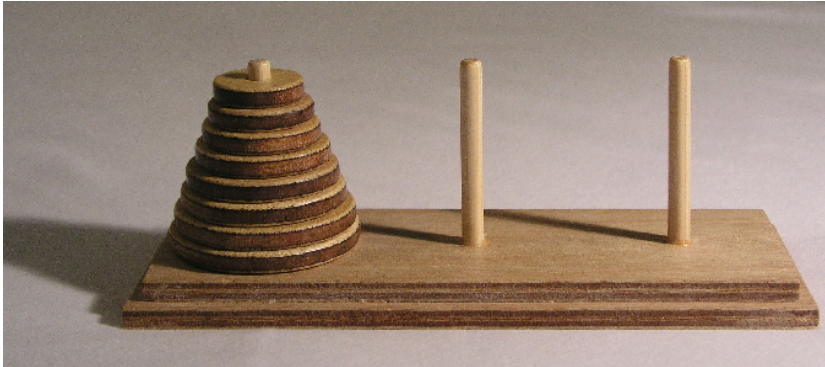
Seed: **-12 000**

- Use a spreadsheet to find the time needed to **pay back** the loan.

52 months

21-The tower of Hanoi

The historical Tower of Hanoi game seems a difficult task at the beginning but, after some time of manipulation, patterns recognition helps us to break down it.



A model set of the Tower of Hanoi (with 8 disks)

1. The game

It consists of three pegs, and a number of disks, each one larger than the one above.

The aim is to move all the disks onto another peg, in the **smallest** possible number of moves. You must follow these rules:

- You can only move one disk at a time.
- You can never put a larger disk on top of a smaller disk.

2. How to play?

Provide you with disks of different sizes. For example, draw circles of increasing radius on cardboard or even regular paper, and cut them out.

The pieces can be stacked on top of each other to simulate the game.



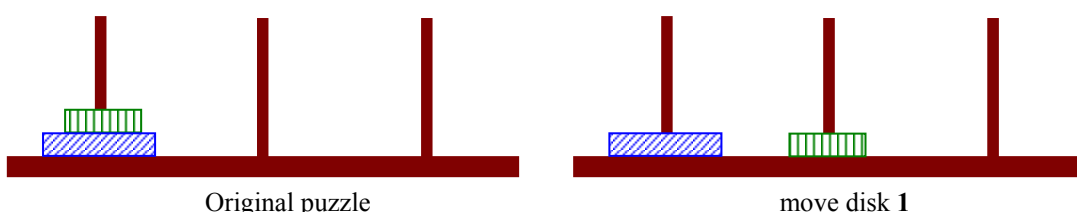
The **best strategy** is to start with simple versions and look for patterns.

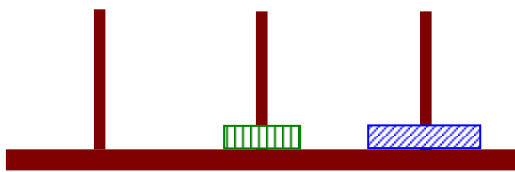
3. Only one disk



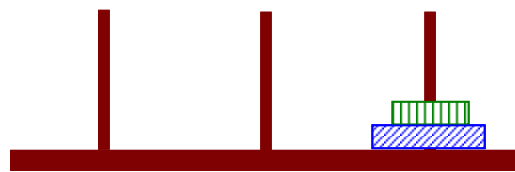
You need only **one** move.

4. Two disks





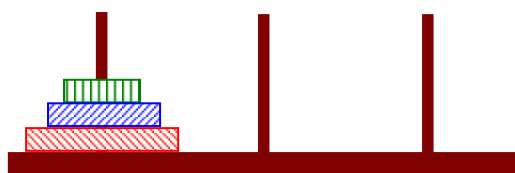
move disk 2



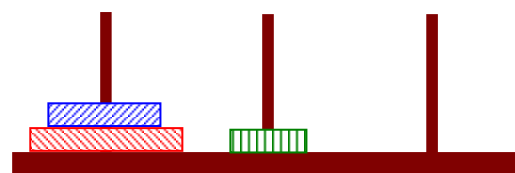
move disk 1

At least **three** moves are required. We can describe the solution with the sequence of numbers **1 2 1** (it gives the order in which the disks must move).

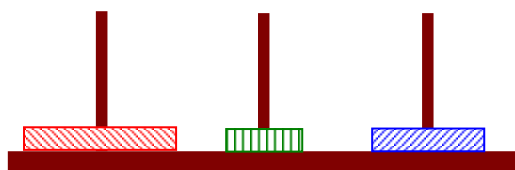
5. Three disks



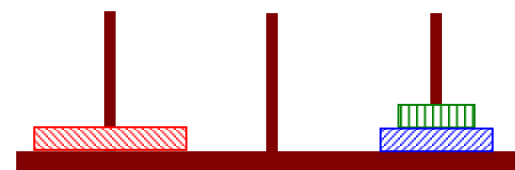
Original puzzle



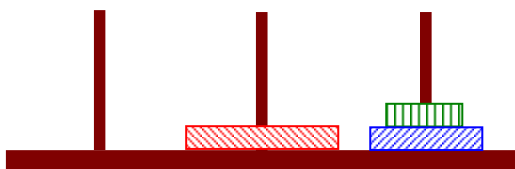
move disk 1



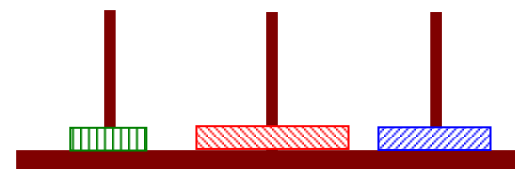
move disk 2



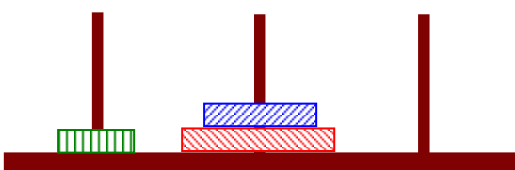
move disk 1



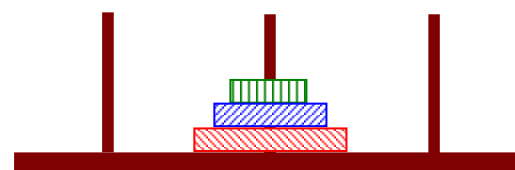
move disk 3



move disk 1



move disk 2



move disk 1

- How many moves do you need to move the three disks onto another peg?

At least seven moves

- Describe your solution by giving the sequence of disks numbered 1, 2 and 3, from the top to the bottom.

1 2 1 3 1 2 1

- Check your answer with your **partners**.
- **Notice** that in the process of solving this puzzle, you **first** solve the two-disk puzzle, **then** move the bottom disk to an empty peg and **finally** solve the puzzle for two disks again.

6. Four disks

First make sure you understand how to solve the puzzle when given only one disk, two disks, or three disks to start with.

Now solve the puzzle for four disks.

Describe the sequence of disks that you must move using 1, 2, 3 and 4, as before.

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

7. The game on the Web

Run simulations of the Tower of Hanoi puzzle, for example at

<http://www.mathsnet.net/puzzles/hanoi/index.html>

8. Five disks

Describe your strategy for completing the five-disk puzzle. Try to use the word **iteration**.

We know to solve the puzzle for four disks, so we move in 15 steps the disks 1, 2, 3 and 4 to another peg. Then we move the disk 5 to the empty peg. Finally we put the disks 1, 2, 3 and 4 on top the disk 5 in 15 steps again.

The solution is given by the sequence:

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 5 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

We followed the iteration rule: solve for 4 disks, move the largest disk, then solve for 4 disks again.

9. Solving the game

List the total number of moves necessary to solve the puzzle for 1 disk, for 2 disks, for 3 disks, for 4 disks and for 5 disks.

1, 3, 7, 15, 31

Regard this is an orbit generated by numerical iteration. Find out:

- Seed **1**
- Iteration rule **Multiply by 2, then add 1** or **$x \rightarrow 2x + 1$**

How many moves do you need to solve the puzzle with 6 disks? **Discuss** with your partner and **explain** how you found this out.

63 moves

Solve for 5 disks, move the larger disk and solve for 5 disks again.

10. Read the text and answer the questions below:

Maybe it is better ask the next questions from the beginning and let pupils to find out themselves the answers in the Internet.

Fascinating Facts

The Tower of Hanoi (sometimes referred to as the Tower of Brahma or the End of the World Puzzle) was invented by the French mathematician, Edouard Lucas, in 1883. He was inspired by a legend that tells of a Hindu temple where the pyramid puzzle might have been used for the mental discipline of young priests. Legend says that at the beginning of time the priests in the temple were given a stack of 64 gold disks, each one a little smaller than the one beneath it. Their assignment was to transfer the 64 disks from one of the three poles to another, with one important proviso: a large disk could never be placed on top of a smaller one. The priests worked very efficiently, day and night. When they finished their work, the myth said, the temple would crumble into dust and the world would vanish.



Temple Pura Ulu Danau, Bali

Where's the Math in this Game?

The number of separate transfers of single disks the priests must make to transfer the tower is 2 to the 64th minus 1, or 18,446,744,073,709,551,615 moves! If the priests worked day and night, making one move every second it would take slightly more than 580 billion years to accomplish the job! You have a great deal fewer disks than 64 here. Can you calculate the number of moves it will take you to move the disks from one of the three poles to another?



<http://www.lhs.berkeley.edu/Java/Tower/tower.html>

- Who invented The Tower of Hanoi puzzle?

The French mathematician Edouard Lucas

- When was it invented?

In 1883

- What was he inspired by?

He was inspired by a Hindu legend

- How many years are a billion of years?

A million millions: 10^{12}

In United States, a thousand millions: 10^9

Be careful about this word. Financial press or Governments use often 10^9 in their reports.

- According to the text the priests need $2^{64} - 1$ moves to transfer 64 disks from one pole to another. **Check** this formula with yours previous solutions of the puzzle. **Argue** about it with your partners and your teacher

1 disk	$2^1 - 1 = 1$ move
2 disks	$2^2 - 1 = 4 - 1 = 3$ moves
3 disks	$2^3 - 1 = 8 - 1 = 7$ moves
4 disks	$2^4 - 1 = 16 - 1 = 15$ moves
5 disks	$2^5 - 1 = 32 - 1 = 31$ moves
6 disks	$2^6 - 1 = 64 - 1 = 63$ moves

The pattern above suggests that the priests need indeed at least $2^{64} - 1$ moves to transfer 64 disks.

LESSON PLAN 2. Geometric Iteration

Aim: To introduce pupils to the process of iteration from a geometric approach, that is, to generate a sequence of figures under a geometric iteration rule and then predict the fate of that sequence.

Teaching objectives:

Content

- Geometric iteration rule
- Seed or initial figure of the iteration
- Orbit of the seed under the iteration rule
- Fate of the orbit
- Areas and perimeters
- Geometric transformations. Rotation.
- Similarity. Scale factor.

Communication

- Predicting: “tends to ...”, “will be...”
- Arguing with partners rules and fates of orbits
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Reading summary sentences to other groups

Cognition

- Identifying iteration rules
- Predicting eventual outcomes for various seeds
- Drawing geometric shapes: straight line segments, squares, triangles and circles
- Calculating scale factors, perimeters and areas
- Creating figures through drawing programs
- Writing summary reports using key vocabulary
- Arguing and checking answers

Context

- Images development
- Zooms of the size of images
- Illustration, graphic design and visual art

Outcomes: At the end of the lesson, pupils will be able to

- Know each step of an iterative geometric process
- Identify geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in geometric English language
- Recognise the importance of arguing in a team
- Know how to behave in a discussion

Key words: geometric iteration rule, seed, orbit, fate, cycle, period.

Prerequisites: Pupils need to be familiar with basic geometric shapes, reduction and enlargement, rotation and midpoints.

Curriculum connections:

- Calculation of perimeters and areas of squares and circles.
- Computation with fractions and square roots in a geometric context.
- Pythagoras' Theorem.
- Triangle's geometry.
- Practise with recognizing and describing geometric patterns.
- Geometric transformations. Similarity.
- Use of calculator or computer when needed.

Timing: Four 55-minute class periods.

Tasks planned:

First session

- The worksheets 1-*Shrinking iteration rules, exercises 1, 2 and 3* will be provided.
- Iteration key words (rule, seed, orbit and fate of orbit) must be familiar from the previous unit. Emphasize now the seed is a figure (not a number) that becomes smaller (*shrinks*) under the “Shrinking iteration rule”.
- Pupils work through these exercises using pencil, ruler and set square.
- Plenary: correction and discussion.
- Show how a computer drawing program allows us to shrink an image by varying percentages to simulate the reductions by $\frac{1}{2}$, $\frac{1}{4}$ and so on.
- Homework: Exploring 1-*Shrinking iteration rules, exercise 4*.

Second session

- Plenary: correction of homework and discussion about it.
- Pupils work through 1-*Shrinking iteration rules, exercise 5* in cooperative groups.
- Plenary: summary and conclusions.
- Homework: Exercise 2-*Rotating iteration rules*

Third session

- Pupils show their homework pictures to their classmates.
- Show how a computer drawing program allows us to rotate an image.
- Pupils work through 3-*Replacement iteration rules* in cooperative groups.
- Plenary: comments on answers of the previous exercise.
- Homework: Exploring 4-*Midpoint iteration rules, exercise 1*.

Fourth session

- Revision of (or maybe introducing) Pythagoras' Theorem.
- Pupils work through 4-*Midpoint iteration rules* in cooperative groups.
- Plenary: summary discussion.
- Homework: Ask pupils to finish the remaining exercises.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For drawings: ruler, compasses, set squares, coloured pencils ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Iteration* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

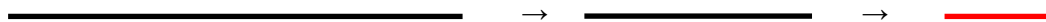
Evaluation

TEACHER’S NOTES AND ANSWERS

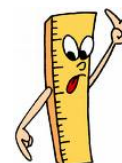
Answers to exercises

1-Shrinking iteration rules

1. The seed is the **straight line segment** below. The iteration rule is to **shrink** the segment so that its length is **half** the length of the original one.



- a) **Draw** above the third segment of the orbit. Use your **ruler**.



- b) Assume the **length** of the original straight line segment is **1 unit**.
-What **fraction** is the length of the second segment?

1 / 2

-What fraction is the length of the third segment?

1 / 4

- c) Draw below the next three segments along the orbit.



- d) What fractions are the lengths of these three segments?

1 / 8 → 1 / 16 → 1 / 32

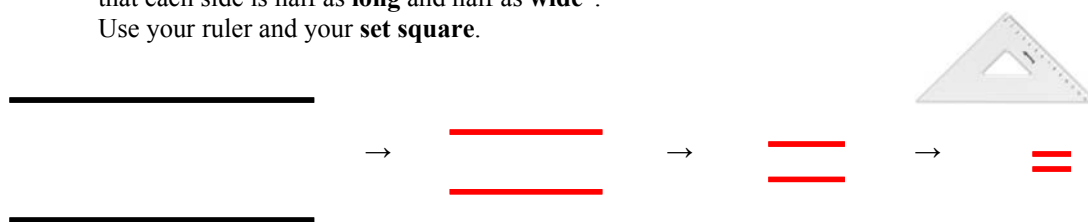
- e) In the grid below **tick** true or false beside each statement.

	True	False
The fate of the orbit is a single point.	✓	
The fate of the orbit is a very small segment.		✓
The orbit tends to zero.		✓
The orbit disappears.		✓
The length of the orbit’s segments tends to zero.	✓	

- f) Check your answers with your partner.

Don’t confuse the shape “straight line segment” with his length.

- a) **Sketch** the next three pictures in the orbit using the iteration rule: “Shrink the previous figure so that each side is half as **long** and half as **wide**”. Use your ruler and your **set square**.



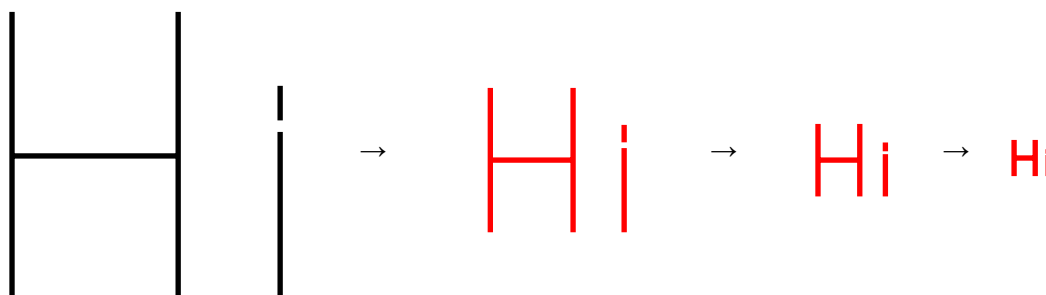
- b) Is the orbit approaching a single point or two points?

The orbit is approaching a single point.

- c) Why?

The figure is reduced in both the horizontal direction and the vertical direction. The two straight line segments come closer and closer as they grow smaller and smaller.

3. The seed is the word “Hi” and the iteration rule is: **Shrink** the word “Hi” by a **scale factor of $\frac{1}{2}$** .



- a) Sketch the next 3 words in the orbit.
 b) Explain the fate of the orbit.

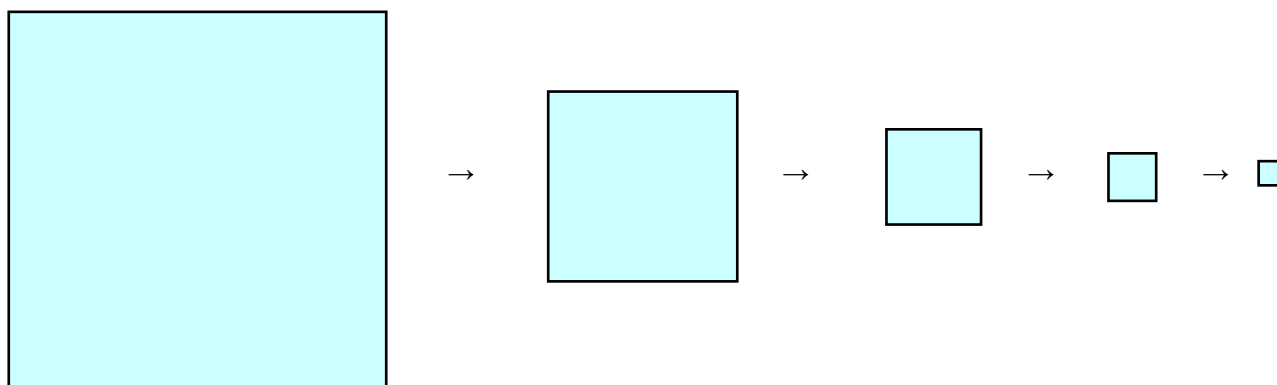
The fate of the orbit is a single point because the two letters get smaller and smaller and closer and closer.

A computer drawing program is useful in helping pupils visualize the shrinking of figures by varying percentages.

In exercise 2 you need previously to group the two straight line segments, make a copy and then apply 50% reduction to get the next picture.

Proceed equally with “H” and ”i”.

4. The seed is the square below. The iteration rule is to shrink the square so that each **side** is half as long.



- a) Draw the next four squares of the orbit.

Pupils use their set drawing or a computer drawing program.

- b) What is the fate of the orbit? **The orbit tends to a single point.**

- c) - Assume the length of each side of the original square is **1 unit**.

- Remember:

- *The **perimeter** of a **square** is the total length of the four sides.*
- *The **area** of a **square** can be found by multiplying the length of two sides.*

- Calculate the length of each side of these four squares, their perimeter and their area and fill in the grid below:

Some pupils measure the side length and don't feel confident what 1 unit is. Others use decimals better than fractions. The teacher knows what each pupil is able to do.

	Side length (as a decimal)	Side length (as a fraction)	Perimeter (in units)	Area (in square units)
Original square	1	1	$4 \cdot 1 = 4$	$1^2 = 1$
First iteration	0.5	$\frac{1}{2}$	$4 \cdot \frac{1}{2} = \frac{4}{2} = 2$	$\left(\frac{1}{2}\right)^2 = \frac{1}{2^2} = \frac{1}{4}$
Second iteration	0.25	$1 / 4$	$4 \cdot (1 / 4) = 1$	$(1 / 4)^2 = 1 / 4^2 = 1 / 16$
Third iteration	0.125	$1 / 8$	$4 \cdot (1 / 8) = 1 / 2$	$(1 / 8)^2 = 1 / 8^2 = 1 / 64$
Fourth iteration	0.0625	$1 / 16$	$4 \cdot (1 / 16) = 1 / 4$	$(1 / 16)^2 = 1 / 16^2 = 1 / 256$

- b) Complete the sentences:

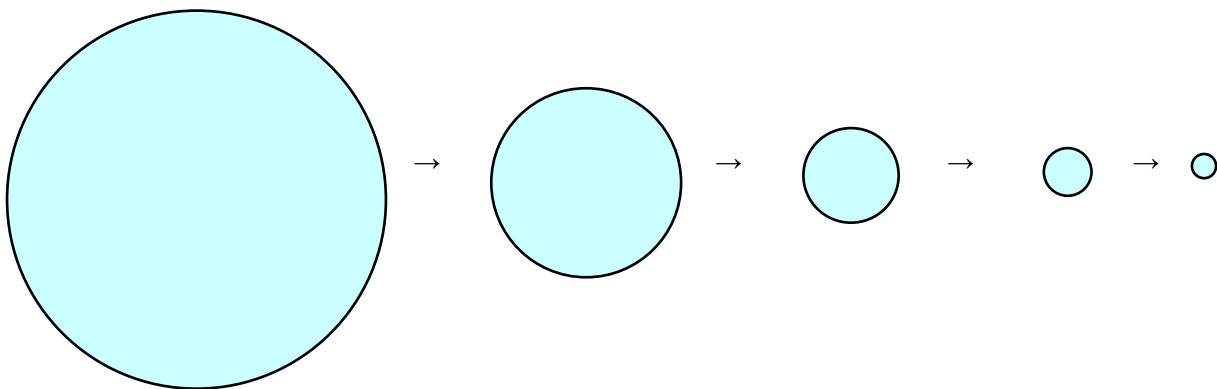
- The side length is decreasing by a scale factor of **$1 / 2$**
- The perimeter is decreasing by a scale factor of **$1 / 2$**
- The area is decreasing by a scale factor of **$1 / 4$**

Pupils should notice the area scale factor being the square of the side length scale factor.

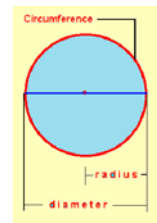
- c) Predict (as a fraction) the side length, the perimeter and the area of the eleventh iteration.
- The side length of the eleventh iteration will be $(1 / 2)^{11} = \underline{1 / 2048}$ units.
 - The perimeter will be $4 \cdot (1 / 2)^{11} = 4 / 2048 = \underline{1 / 512}$ units.
 - The area will be $((1 / 2)^{11})^2 = (1 / 2048)^2 = \underline{1 / 4\ 194\ 304}$ square units or, using the scale factor, $(1 / 4)^{11} = 1 / 4\ 194\ 304$ square units.

Maybe some pupils feel more confident listing the fifth iteration, the sixth and so on.

5. The seed is the **circle** below and the iteration rule is to shrink the circle so that the **diameter** is half as long.



- a) Draw the next four circles of the orbit.
 b) What is the fate of the orbit?
The orbit tends to a single point.
- c) - Assume the diameter's length of the original circle is **2 units**.
 - Remember:
Circumference = $2 \cdot \pi \cdot \text{radius} = \pi \cdot \text{diameter}$
Circle Area = $\pi \cdot r^2$
- Complete the table below:



	Diameter	Radius	Perimeter (in units)	Area (in square units)
Original circle	2	1	$\pi \cdot 2 = 2\pi$	$\pi \cdot 1^2 = \pi \cdot 1 = \pi$
First iteration	1	$\frac{1}{2}$	$\pi \cdot 1 = \pi$	$\pi \cdot \left(\frac{1}{2}\right)^2 = \pi \cdot \frac{1}{2^2} = \frac{\pi}{4}$
Second iteration	1 / 2	1 / 4	$\pi \cdot 1 / 2 = \pi / 2$	$\pi \cdot (1 / 4)^2 = \pi / 4^2 = \pi / 16$
Third iteration	1 / 4	1 / 8	$\pi \cdot 1 / 4 = \pi / 4$	$\pi \cdot (1 / 8)^2 = \pi / 8^2 = \pi / 64$
Fourth iteration	1 / 8	1 / 16	$\pi \cdot 1 / 8 = \pi / 8$	$\pi \cdot (1 / 16)^2 = \pi / 16^2 = \pi / 256$

d) Complete the sentences:

- The diameter and the radius are decreasing by a scale factor of $1/2$
- The perimeter is decreasing by a scale factor of $1/2$
- The area is decreasing by a scale factor of $1/4$

e) Predict (as a fraction) the diameter, the radius, the perimeter and the area of the eleventh iteration.

- The diameter of the eleventh iteration will be $(1/2)^{10} = 1/1024$ units.
- The radius will be $(1/2)^{11} = 1/2048$ units.
- The perimeter will be $\pi \cdot (1/2)^{10} = \pi/1024$ units.
- The area will be $\pi \cdot ((1/2)^{11})^2 = \pi / (1/2048)^2 = \pi / 4\,194\,304$ square units, or, using the scale factor $\pi \cdot (1/4)^{11} = 1/4\,194\,304$ square units.

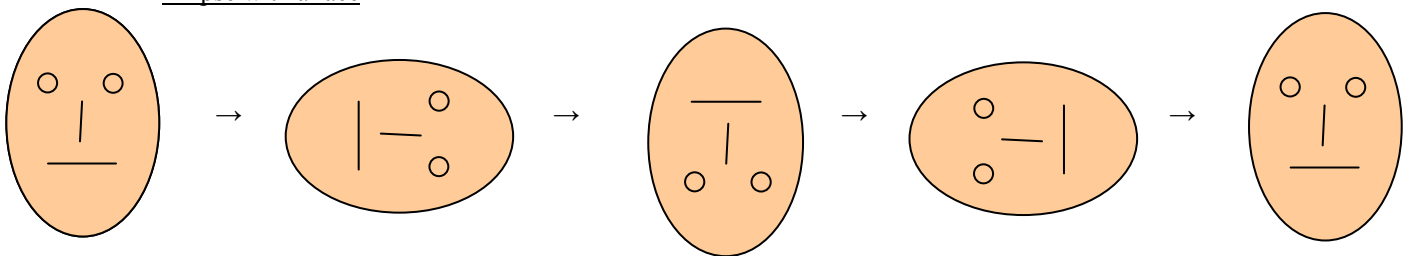
2-Rotating iteration rules

1. Using the iteration rule “Rotate the figure 90° clockwise”, draw down the next four pictures in the orbits for the following seeds:

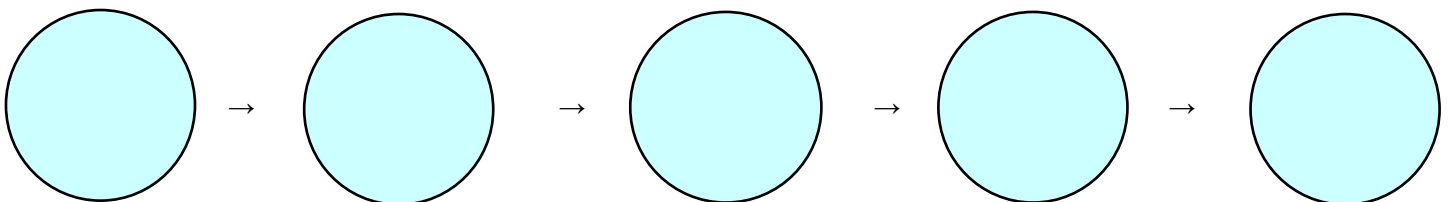
Pupils use their set drawing or a computer drawing program. You can ask to draw other different pictures and to rotate them.

Discuss about “clockwise” and the opposite “counter-clockwise”.

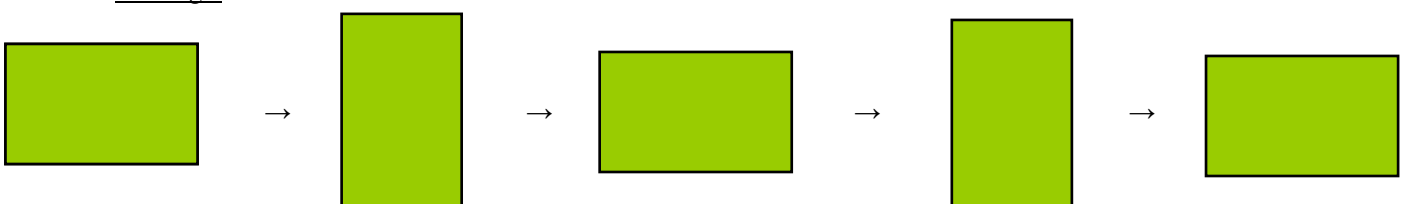
Ellipse with a face



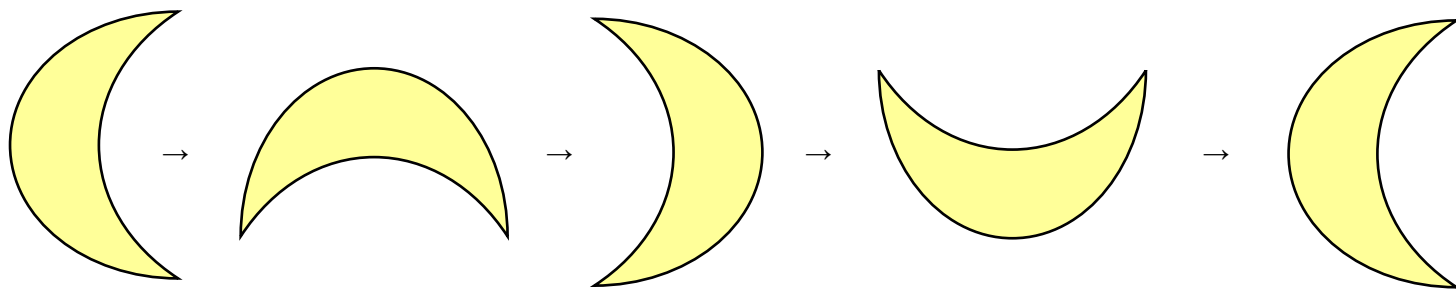
Circle



Rectangle



Waning Moon



2. Describe the fate of these orbits.

- **The orbit of the face is a cycle of period 4.**
- **The orbit of the circle stays fixed.**

Actually the points inside the circle are rotating, but the picture always looks like the same circle.

- **The orbit of the rectangle is a cycle of period 2.**

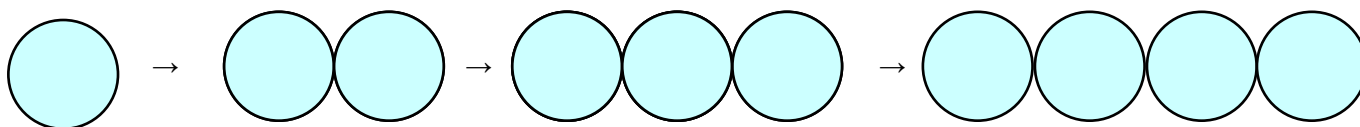
The points inside the rectangle (except the center) return to itself after 4 iterations, but the picture itself is the same after 2 iterations.

- **The orbit of the moon is a cycle of period 4.**

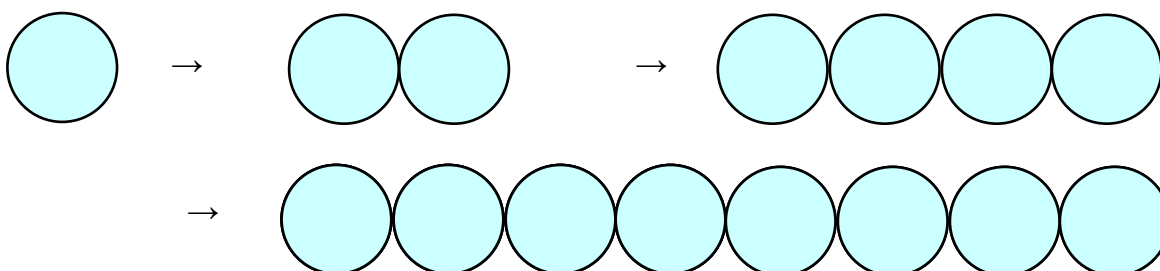
3-Replacement iteration rules

1. Using the seed “the circle below”, draw down the next three pictures in the orbits for the following iteration rules:

Place a circle of the same size just to the right of the previous circle



Place an exact copy of the previous figure just to the right of the previous figure



b) How many circles you will have in the tenth iteration?

First rule: **2 circles**

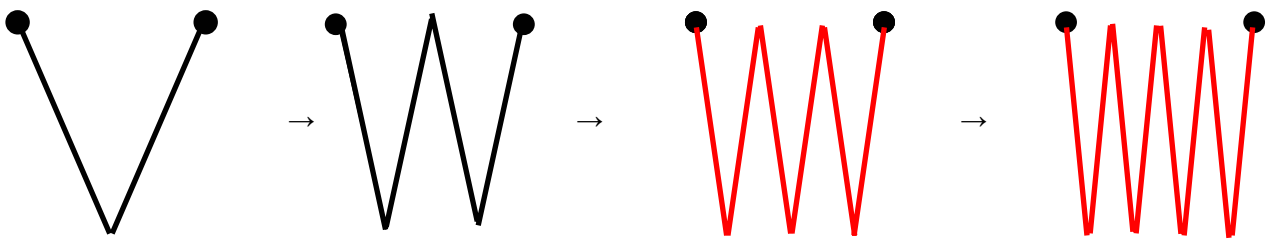
Second rule: **$2^{10} = 1\,024$ circles**

c) **Describe** the fate of these orbits.

The fate of these orbits is an infinitely long string of circles, all copies of the original circle.

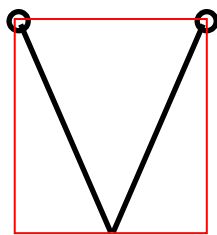
Point out the fate of these orbits are the same, even the number of circles grows much more quickly under the second iteration rule than under the first one.

2. a) Follow the iteration:



This orbit is easy to sketch using a paint program that resizes and regroups images.

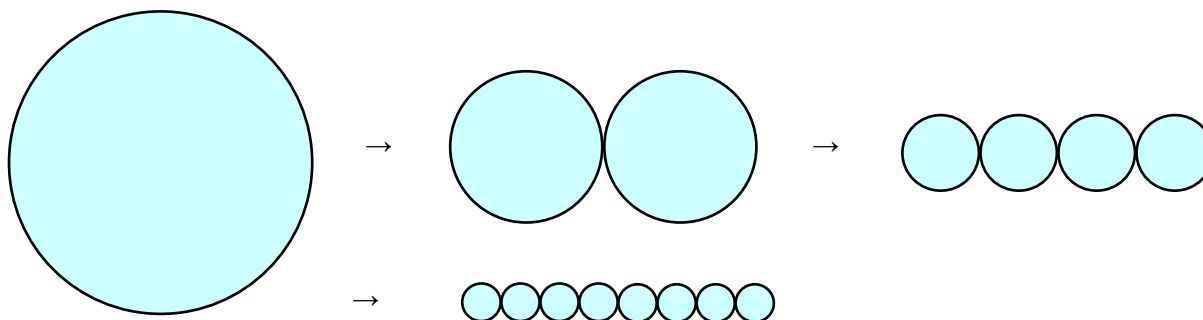
b) What **shape** is the fate of this orbit?



The fate of the orbit is a filled rectangle, because the straight line segments become more and more closely packed.

Pay attention that any figure on this orbit contains two vertical segments. The sequence of pictures approaches the filled rectangle; we don't get a final figure.

3. The seed is the **circle** below and the iteration rule is to **replace** a circle by **two smaller copies** of itself (lined up side by side) whose diameters are each one half of the original



- a) Draw the next three pictures in the orbit.
- b) How many circles you will have in the ninth iteration? $2^9 = \underline{512}$ circles
- c) What shape is the orbit approaching?

The orbit is approaching a straight line segment whose length is the diameter of the original circle.

- d) - Assume the diameter's length is **2 units**.
- Complete the table below:

	Perimeter (in units)	Area (in square units)
Original circle	$\pi \cdot 2 = 2\pi$	$\pi \cdot 1^2 = \pi \cdot 1 = \pi$
First iteration	$2 \cdot \pi \cdot 1 = 2\pi$	$2 \cdot \pi \cdot \left(\frac{1}{2}\right)^2 = 2 \cdot \pi \cdot \frac{1}{2^2} = \frac{\pi}{2}$
Second iteration	$4 \cdot \pi \cdot \frac{1}{2} = 2\pi$	$4 \cdot \pi \cdot (1/4)^2 = \pi / 4$
Third iteration	$8 \cdot \pi \cdot \frac{1}{4} = 2\pi$	$8 \cdot \pi \cdot (1/8)^2 = \pi / 8$

- e) Predict the perimeter and the area of the ninth iteration.

The perimeter of the ninth iteration will be 2π units.

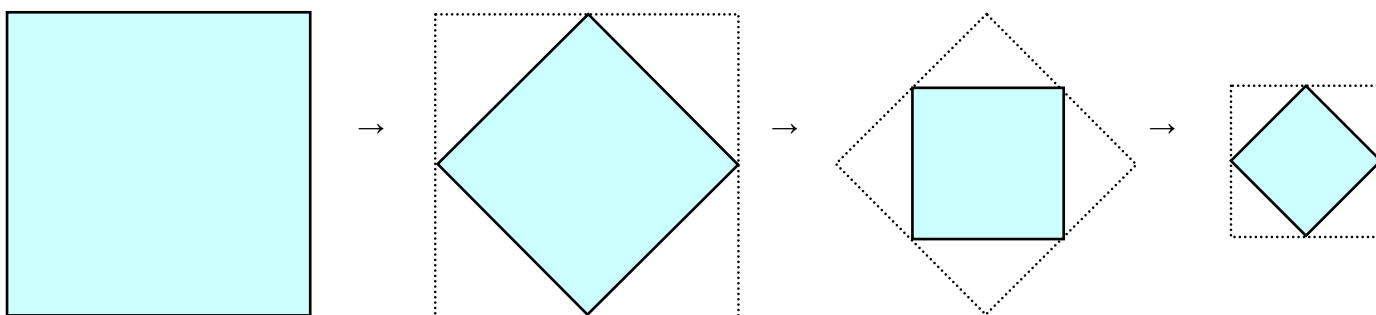
The area of the second iteration is $\pi / 4 = \pi / 2^2$ square units and the area of the third iteration is $\pi / 8 = \pi / 2^3$ square units, so the area of the ninth iteration will be $\pi / 2^9 = \pi / 256$ square units

4-Midpoint iteration rules

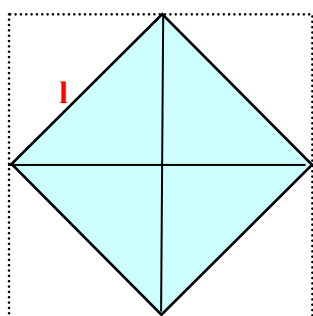
1. The seed is the **filled** square below. The iteration rule is:
 - Sketch the **boundary** of the previous square.
 - Find the **midpoint** of each side.
 - Connect **adjacent** midpoints with a straight line segment to form a new square.
 - **Remove** the old square and **shade** the interior of the new square.

a) Draw the next three pictures in the orbit.

Pupils can find the midpoints carefully by measuring. Using a drawing program they make a copy of the original square, apply 50% reduction and then a 90° rotation to get the next picture.



b) Express the area of these three squares as a percentage of the area of the original square.



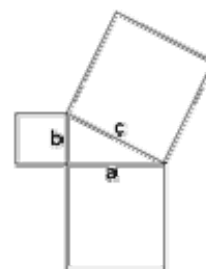
50% → 25% → 12.5%

c) What is the fate of the orbit?

The orbit tends to a single point.

d) Assume the length of each side of the original square is **2 units**.
Remember

Pythagoras' Theorem:
In a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.



If pupils are not familiar with Pythagoras' Theorem, it is a good moment to mention the relationship in a right angled triangle between the longest side and the other two sides.

Pupils non confident with square roots can use decimals.

Others skip over the length of 2 units and just measure using their ruler.

- Calculate the side length of the first iteration.

$$l^2 = 1^2 + 1^2$$

$$l = \sqrt{1^2 + 1^2} = \underline{\sqrt{2}} \text{ units}$$

- Calculate the side length of the second iteration.

$$l^2 = (\sqrt{2}/2)^2 + (\sqrt{2}/2)^2$$

$$l = \sqrt{2/4 + 2/4} = \sqrt{4/4} = \underline{1} \text{ unit}$$

- Calculate the side length of the third iteration.

$$l^2 = (1/2)^2 + (1/2)^2$$

$$l = \sqrt{1/4 + 1/4} = \sqrt{2/4} = \underline{1/\sqrt{2}} \text{ units}$$

- Complete the table below.

	Side length (in units)	Perimeter (in units)	Area (in square units)
Original square	2	$4 \cdot 2 = 8$	$2^2 = 4$
First iteration	$\sqrt{2}$	$4 \cdot \sqrt{2} = 4\sqrt{2}$	$(\sqrt{2})^2 = 2$
Second iteration	1	$4 \cdot 1 = 4$	$1^2 = 1$
Third iteration	$1/\sqrt{2}$	$4 \cdot (1/\sqrt{2}) = 4/\sqrt{2}$	$(1/\sqrt{2})^2 = 1/2$

- Compare the area founded to your answer b)

The area is decreasing by a scale factor of 1 / 2.

e) Predict the area of the sixth iteration.

The area of the sixth iteration will be 1 / 16 square units.

f) Complete the sentences:

- The area is decreasing by a scale factor of **1 / 2**

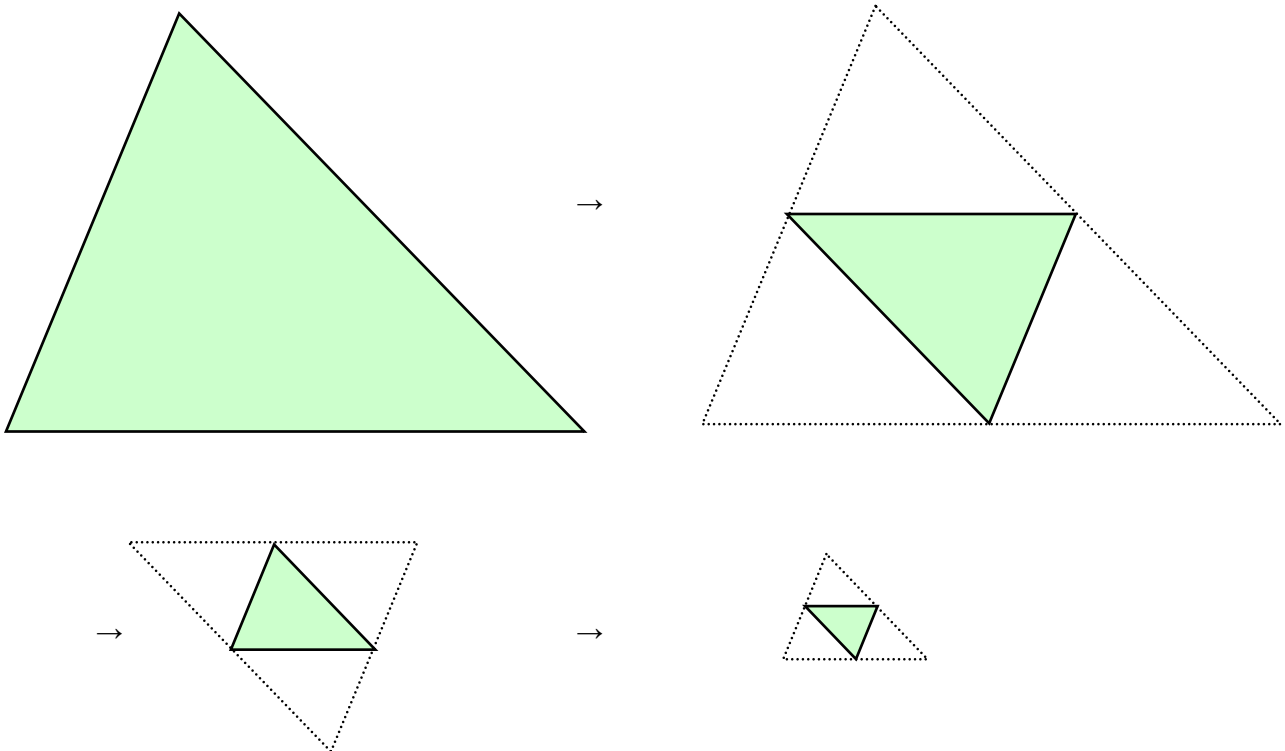
- The side length is decreasing by a scale factor of **$\sqrt{1/2} = 1/\sqrt{2}$**

g) Predict the side length of the sixth iteration.

The side length of the sixth iteration will be $\sqrt{1/16} = 1/4$ units.

2. The seed is **any** triangle and the iteration rule “**Join** the midpoints of each of the sides of the triangle to form a new triangle”.

a) Draw the next three pictures in the orbit.



b) Express the area of these three triangles as a percentage of the area of the original triangle.

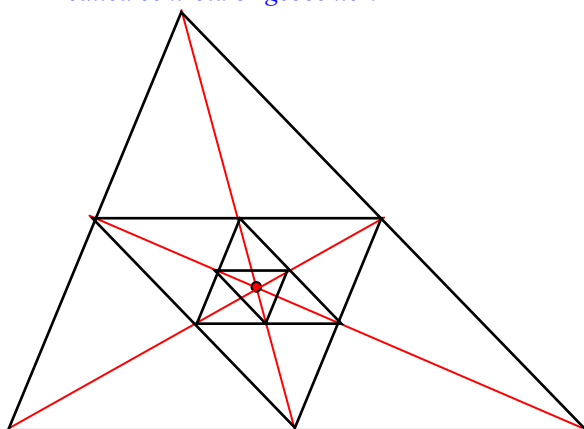
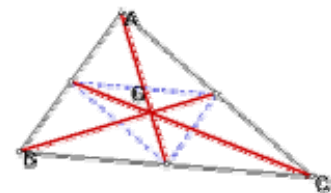
50% → 25% → 12.5%

c) What is the fate of the orbit? **The orbit tends to a single point.**

d) Draw the **medians** of the original triangle and note how they **intersect** each subsequent triangle on the orbit

Remember

- The medians of a triangle are segments from each vertex to the midpoint of the opposite side.
- The medians always intersect in a single point, called *centroid* or *geocenter*.



e) Write down your **conclusion** by completing the sentence below.

The fate of the orbit is **actually the intersection point of the medians, called centroid or geocenter.**

LESSON PLAN 3. The Sierpiński triangle

Aim: To introduce pupils to one of the simplest examples of the geometric objects known as fractals, created by following a specific set of rules. Rules involve dividing an image into smaller pieces similar to the original and then removing some of those pieces. One of the formal properties of fractals, self-similarity, appears.

Teaching objectives:

Content

- Geometric iteration rule by removals
- Copies of copies method
- Areas and perimeters
- Numerical and geometric patterns
- Geometric transformations. Central symmetry
- Self-similarity

Communication

- Predicting: “tends to ...”, “will be...”
- Comparing: “getting smaller ...”, “getting larger ...”, “magnified by a factor of ...”, “reduced by ...”
- Describing your work: “I have chosen ...”, “I have reduced ...”, ...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Sharing opinions to organise a poster
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition

- Identifying similar shapes
- Predicting eventual outcomes
- Calculating parts of a unit whole, perimeters and areas
- Creating figures through drawing programs
- Creating spreadsheets to display data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers
- Using Internet resources to confirm predictions

Context

- Zooms of the size of images
- Computer graphics and visual art
- The chaos game
- History of Mathematics

Outcomes: At the end of the lesson, pupils will be able to

- Distinguish between two similar figures and a self-similar figure
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Confirm predictions using the Internet resources
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Know how to make a script for an oral presentation
- Present their work to the rest of the class

Key words: geometric iteration rule, infinity, Sierpiński triangle, self-similarity, Pascal triangle, Chaos game.

Prerequisites: Pupils need to be familiar with equilateral triangles, regular tetrahedra, whole powers and fractions as a part of a whole.

Curriculum connections:

- Calculation of perimeters and areas.
- Computation with fractions in a geometric context.
- Triangle's geometry.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Use of calculator or computer when needed.
- Pascal triangle.
- Tetrahedra.

Timing: Nine 55-minute class periods.

Tasks planned:

First session

- Pupils work through exercise 1-*Removals* using pencil, ruler and coloured pencils. A geometric iteration rule must be familiar from the previous unit, so have them practise on their own. Only circulate among them to be sure they are headed in the right direction.
- Don't correct their predictions. Pupils can do it themselves after having sketched the fourth iteration.
- Plenary: sharing and discussing solutions.
- Homework: Ask to finish accurately the exercise.

Second session

- Provide pupils with worksheet 2- *Sierpiński*. They go to the indicated Web site and look carefully at the eighth iteration of the first session exercise.

- Pupils work through this exercise. Emphasize those iterations are approaching the Sierpiński triangle: none of them is the Sierpiński triangle itself.
- Ask pupils to start working with exercise 3-*Area of the Sierpiński triangle* and to finish it as homework.

Third session

- Plenary: correction of exercise 3 and discussion about it.
- Pupils work through exercise 4-*Perimeter of the Sierpiński triangle*.
- Plenary: correction of exercise 4 and discussion about it.

Fourth session

- Pupils work through exercise 5-*Copies of copies* by using a computer drawing program, familiar from the previous lesson, to sketch again the first four iterations approaching the Sierpiński triangle.
- All of the class cooperates by making a poster following the instructions from exercise 6-*Poster presentation*.
- Homework: Ask pupils to answer the questions from exercise 5.

Fifth session

- Divide the class in half and have one group of pupils working through exercise 7-*Triplicating a circle* and the other group through exercise 8-*Triplicating a square*.
- Each group prepares a poster and writes a script for an oral presentation.
- Plenary: poster presentation and oral process explanation.
- Homework: Ask pupils go one step further by *Triplicating any image* (for example, their own picture) and making a poster with it.

Sixth session

- Each pupil presents to the rest of the class his/her home project.
- Provide pupils with worksheet 10-*Self-similarity* and remain them of the definition of similar figures and what is meant by a ratio of similarity.
- Ask pupils to identify self-similar figures and magnification factors through exercise 10.
- Homework: exercise 11-*Pascal triangle* shows the relationship between the Pascal triangle and the Sierpiński triangle.

Seventh session

- Homework revision.
- Give time to understand the random game in exercise 12-*The Chaos game* and to provide a half-ruler to play hands on.
- Pupils play the game in pairs.
- Homework: running simulations of the game on the Web site.

Eighth session

- All the amount of information from all the pupils is showed altogether.
- Go through *Playing the game with the Sierpiński triangle showing*.
- Plenary: correction of the exercise and comments on it.
- Final conclusion: teacher helps pupils to summarise how a random process can generate a non- random pattern.

Ninth session

- Provide pupils with worksheet 13-*The Sierpiński Tetrahedron*, ice-cream sticks and glue.
- Focus this session on the construction of a Sierpiński Tetrahedron as big as possible. All of the class takes part in it.
- The questions of the exercise must be answered meanwhile.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For drawings and making posters: ruler, pencil and rubber, coloured pencils, cardboard, scissors and glue, half-ruler, pen, markers ...
- For building a *Sierpiński Tetrahedron*: ice-cream sticks and glue.
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Posters will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Answers to exercises

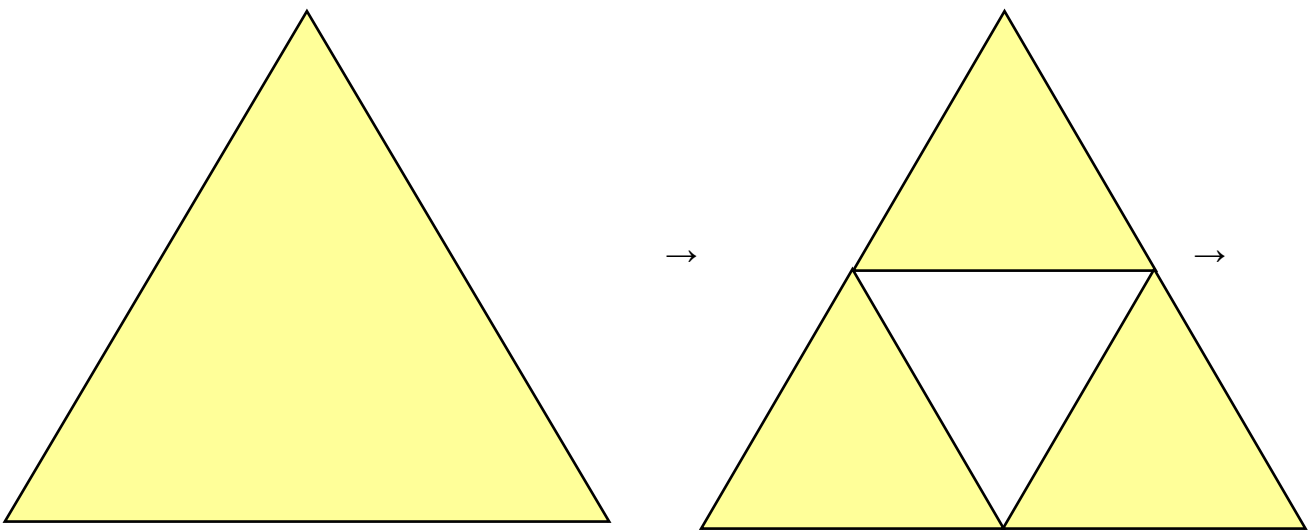
1-Removals

Start with a filled equilateral triangle

Apply the iteration rule:

- Divide it into four equilateral triangles by marking the midpoints of all three sides and drawing lines to connect the midpoints.
- **Remove the interior** of the triangle from the middle.

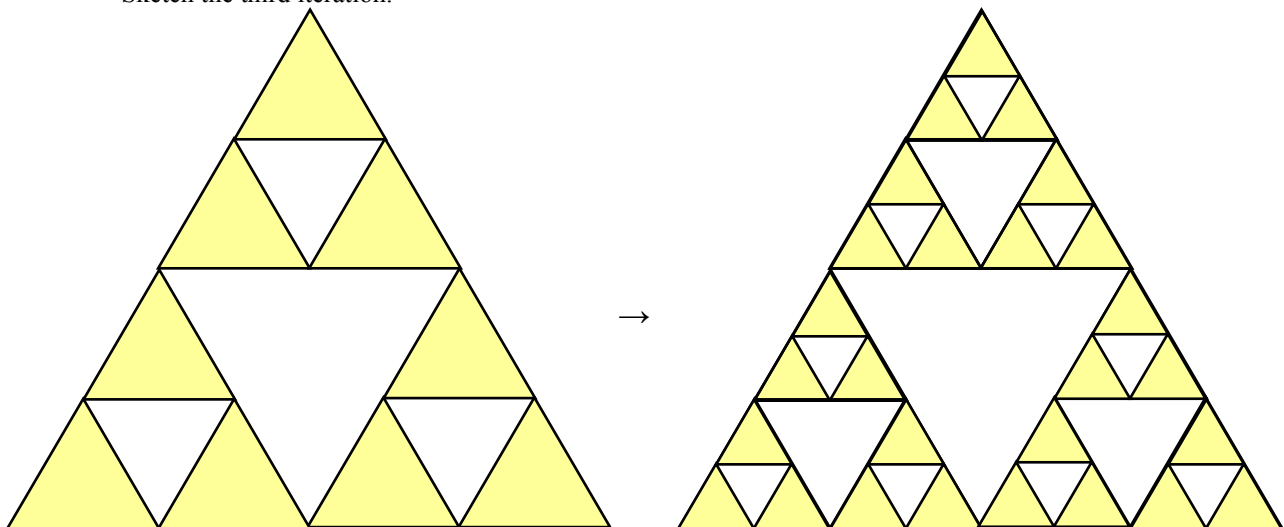
1. Sketch the first iteration. You will have one **hole** and three smaller **filled** equilateral triangle **remaining**.



First iteration

2. - Sketch the second iteration. You will have to apply three times the iteration rule, once on each of the smaller triangles.

- Sketch the third iteration.



Second iteration

Third iteration

3. - **Look carefully at** your sketches and complete the grid below:

	Number of holes added	Total number of holes	Total number of triangles remaining
Original square	0	0	1
First iteration	1	1	3
Second iteration	3	1 + 3 = 4	3 · 3 = 9
Third iteration	9	4 + 9 = 13	9 · 3 = 27

- Check your answers with your partner.

4. Predict the next row of the grid:

If some pupils are wrong about predictions, don't correct them at this step. After drawing the fourth iteration at the end of the exercise, they can go back to this point.

Fourth iteration	27	13 + 27 = 40	27 · 3 = 81
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5. What fraction of the triangle is the total number of triangles remaining in the first iteration?

$$\frac{3}{4}$$

6. What fraction of the triangle is the total number of triangles remaining in the second iteration? **Pay attention** that holes have different sizes.

$$\frac{9}{16}$$

7. What fraction of the triangle is the total number of triangles remaining in the third iteration?

$$\frac{27}{64}$$

8. Do you see a **pattern** here? Use this pattern to predict what fraction of the triangle **would be** the total number of triangles remaining in the fourth iteration.

$$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

$$\underline{\underline{\left(\frac{3}{4}\right)^4 = \frac{81}{256}}}$$

9. Write the fractions in the above questions in order from **greatest to least**.

$$\frac{3}{4} < \left(\frac{3}{4}\right)^2 < \left(\frac{3}{4}\right)^3 < \left(\frac{3}{4}\right)^4$$

10. Assume the **side length** of the original triangle is **1 unit**.

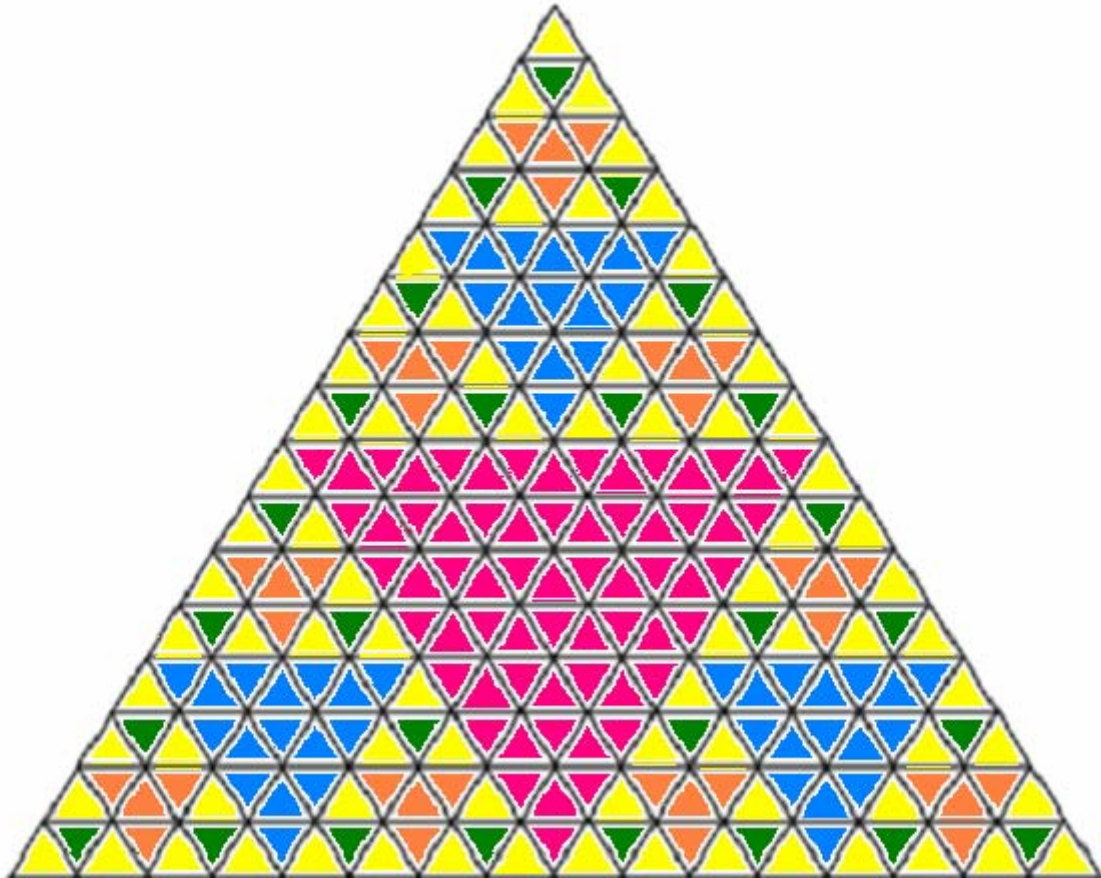
- **Look carefully at** your sketches and complete the grid below:

	Original triangle	First iteration	Second iteration	Third iteration
Side length of each triangle remaining (as a fraction)	1	1 / 2	1 / 4	1 / 8

- What **fraction** would be the side length of each triangle remaining on the fourth iteration?

$$\frac{1}{16}$$

11. Use coloured pencils to sketch below how the fourth iteration looks like.



12. **Confirm** predictions you made before about

- total number of holes and total number of triangles remaining. (4) **40 i 81**
- side length of each triangle remaining. (10) **1 / 16**

2-Sierpiński

1. Go to: <http://math.rice.edu/~lanius/fractals/sierjava.html>



a) Play the game and move from one iteration to the next. The number of holes at each iteration is shown in the corner. **Stop at the 8th** iteration (the number of triangles increases rapidly and a big amount of memory is needed).

b) Fill in the table below:

Iteration	1	2	3	4	5	6	7	8
Total number of holes	1	4	13	40	121	364	1093	3280

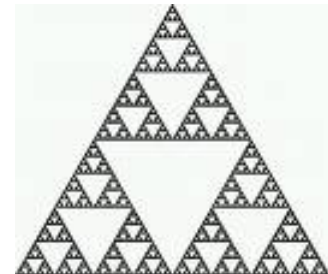
c) Do you see a **pattern** here? (Remember **exercise 1. (3) (4)** above) Use this pattern to **predict** the total number of holes in the 9th and in the 10th iteration.

$1 + 3 = 4$ $4 + 3^2 = 13$ $13 + 3^3 = 40$ $40 + 3^4 = 121$ $121 + 3^5 = 364$

$364 + 3^6 = 1093$ $1093 + 3^7 = 3280$ $3280 + 3^8 = \underline{9841}$ $9841 + 3^9 = \underline{29\ 524}$

d) In your imagination, you could repeat this rule infinitely many times. The fate of this orbit is a shape known as

the Sierpiński triangle fractal.



2. Search for information on the Web and complete the following passage by filling in the blanks



Waclaw Sierpiński (**1882-1969**) was

born in **Warsaw**, the capital of **Poland**.

He described his triangle in the year **1915**.

In 1920 he founded the mathematical journal

Fundamenta Mathematicae, still edited

nowadays by the Polish Academy of Science.

3-Area of the Sierpiński triangle



Assume the **area** of the original triangle is **1 square unit**.

a) Look again at exercise 1 above and complete the grid below (round to two decimal places):

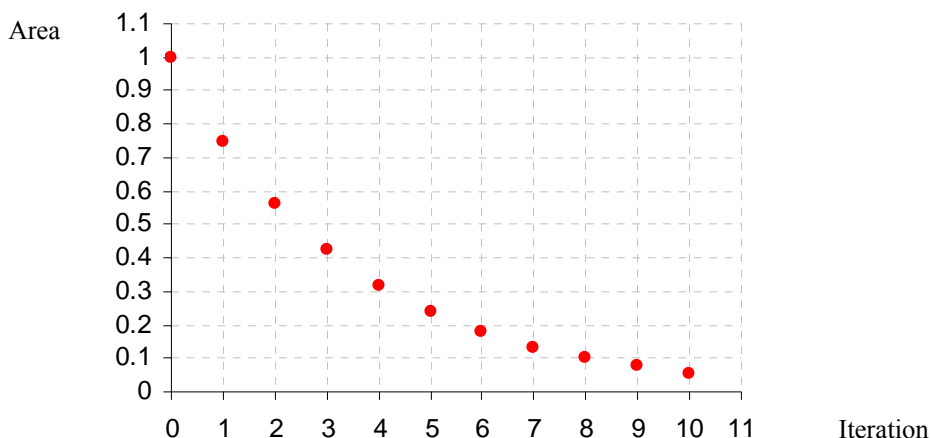
	Original triangle	First iteration	Second iteration	Third iteration	Fourth iteration
Area as a fraction (in square units)	1	$\frac{3}{4}$	$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$	$(3 / 4)^3 = 27 / 64$	$(3 / 4)^4 = 81 / 256$
Area as a decimal (in square units)	1	0.75	0.56	0.42	0.32

b) Use a spreadsheet or calculator to compute the area of the next six iterations.

Enter the seed 1 into the first cell A1 and the formula = A1*0.75, into the second cell A2.

0.24, 0.18, 0.13, 0.10, 0.08, 0.06 rounded to 2 d.p.

c) Sketch a plot of these eleven points.



d) What can you explain about the area if we continue iterating?

The area is getting smaller and smaller because we are adding holes indefinitely.

e) Complete the sentence below:

The area of the Sierpiński triangle is **zero**.

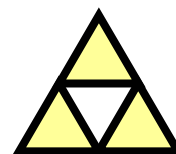
Point out that any particularly iteration has area zero. The statement is about the fate of the orbit.

4-Perimeter of the Sierpiński triangle

- Remember the iteration rule:

“Remove the **interior** of the middle triangle”.

So the **boundary** of the middle triangle **makes up part** of the **perimeter**.



- Assume the **side length** of the original triangle is **1 unit**.

a) Look again at exercise 1 above and complete the grid below:

	Original triangle	First iteration	Second iteration	Third iteration
Side length of each triangle remaining	1	$\frac{1}{2}$	$\frac{1}{4}$	1 / 8
Perimeter of each triangle remaining	$3 \cdot 1 = 3$	$3 \cdot \frac{1}{2} = \frac{3}{2}$	$3 \cdot 1 / 4 = 3 / 4$	$3 \cdot 1 / 8 = 3 / 8$
Total number of triangles remaining	1	3	9	27

Perimeter	$3 \cdot 1 = 3$	$(3/2) \cdot 3 = 9/2$	$(3/4) \cdot 9 = 27/4$	$(3/8) \cdot 27 = 81/8$
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b) Do you see a **pattern** here? Use a spreadsheet or calculator to compute the perimeter of the next seven iterations.

$3^2 / 2 = 9 / 2$

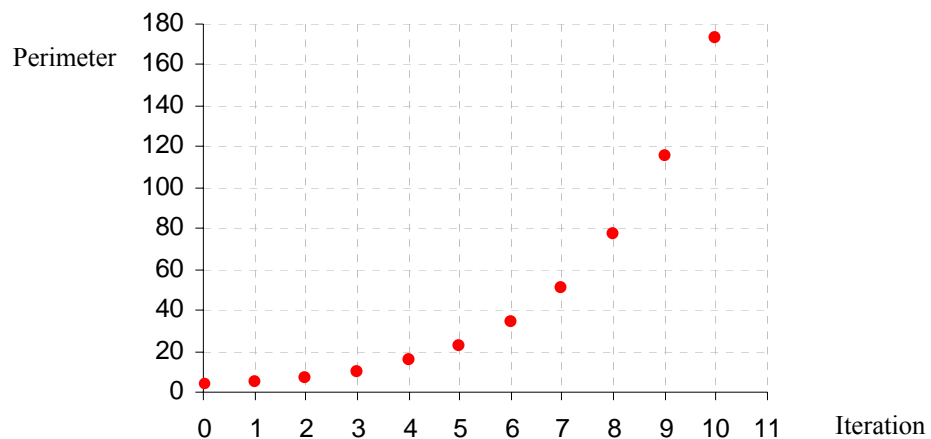
$3^3 / 2^2 = 27 / 4$

$3^4 / 2^3 = 81 / 8$

Enter the seed 3 into the first cell A1 and the formula = A1*1.5, into the second cell A2.

3, 4.5, 6.75, 10.13, 15.19, 22.78, 34.17, 51.26, 76.89, 115.33, 172.96 rounded to 2 d.p.

c) Sketch a plot of these eleven points.



d) What's happening to the perimeter of the Sierpiński triangle?

The perimeter is getting larger and larger. It tends to infinity.

The Sierpiński triangle has area zero and perimeter infinity!!!

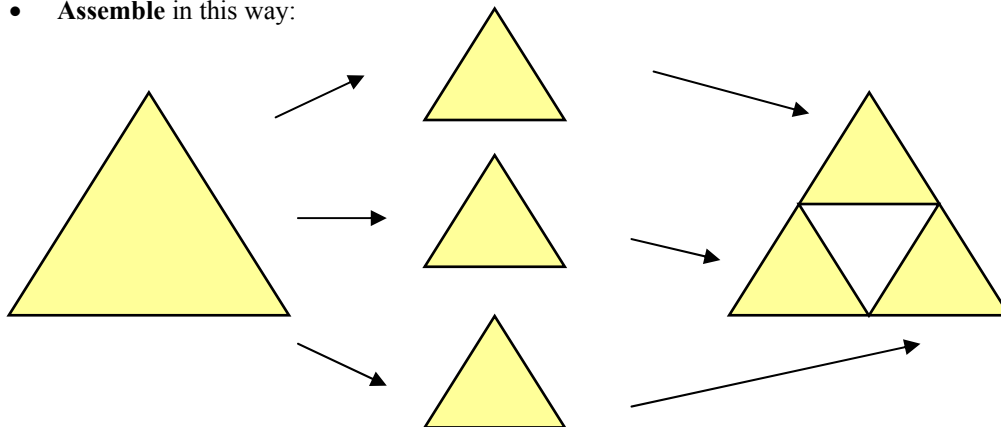
5-Copies of copies

Use any computer **drawing program** in this exercise.

Start **again** with a filled equilateral triangle. **Everyone** in class should start with the **same large** triangle.

The iteration rule is:

- Make three **copies** of the triangle.
- **Reduce** them by 50%.
- **Assemble** in this way:



Now draw the second, the third and the fourth iteration.

What shape will you get by continuing this process? **The Sierpiński triangle**

6-Poster presentation

- **Print** the fourth iteration and **cut out** the triangle.



- **Cooperate** with your classmates by making a **poster** of the **largest** Sierpiński triangle you can create:

- **Assemble** all of the triangles by **matching** their vertices.

- Ask the following questions:

- How many pupils are there in your class? **Open answer**
- How many triangles did you use by making the poster? **Open answer**
- How many copies do you need to be sure you get a Sierpiński triangle? List the possibilities and explain.

1, 3, 9, 27, 81.....We make 3 copies of each copy.

- If each original triangle has sides of length 15 cm, how many copies would you need to produce a Sierpiński triangle whose **base** is 60 cm long?

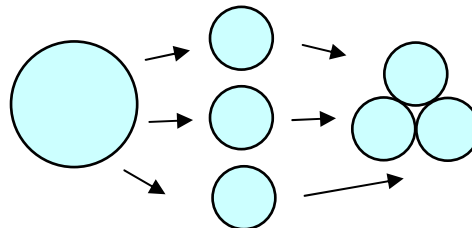
Sixty divide by fifteen is 4, so we need 4 copies on the base, 2 + 2 in the middle and 1 on the top. That is 9 copies.

7-Triplicating a circle

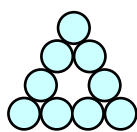


Start with a filled circle and follow the rule:

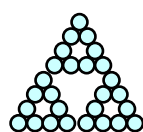
- Make three **copies** of the circle.
- **Reduce** them by 50%.
- **Assemble** in this way:



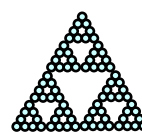
a) Sketch the next three iterations of this rule above.



Second iteration



Third iteration



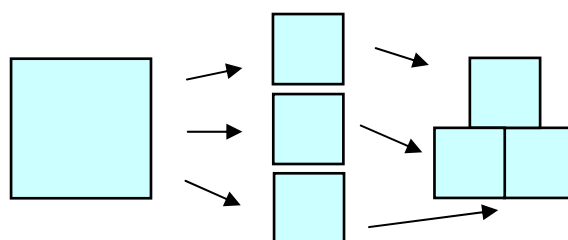
Fourth iteration

b) What figure will result if you continue to iterate this rule? **The Sierpiński triangle**

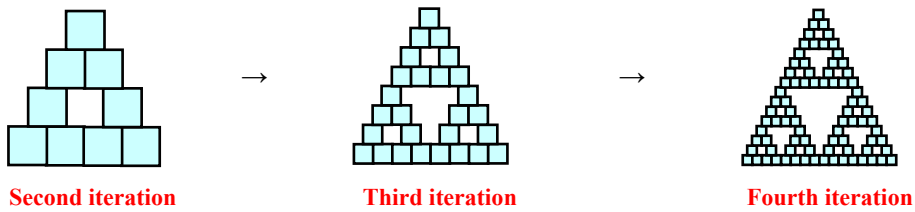
8-Triplicating a square

Start with a filled square and follow the rule:

- Make three **copies** of the square.
- **Reduce** them by 50%.
- **Assemble** in this way:



a) Sketch the next three iterations of this rule above.



b) What figure will result if you continue to iterate this rule? **The Sierpiński triangle**

9-Further exploration



Pick your own photo as your seed or any image you like and proceed with the rule as above. **Organise** your results as a poster and **describe** your work using the past “I have ...”. **Include** your prediction of what you think will result if you continue to iterate this rule indefinitely.

I have chosen a photo of mine and I have reduced it.

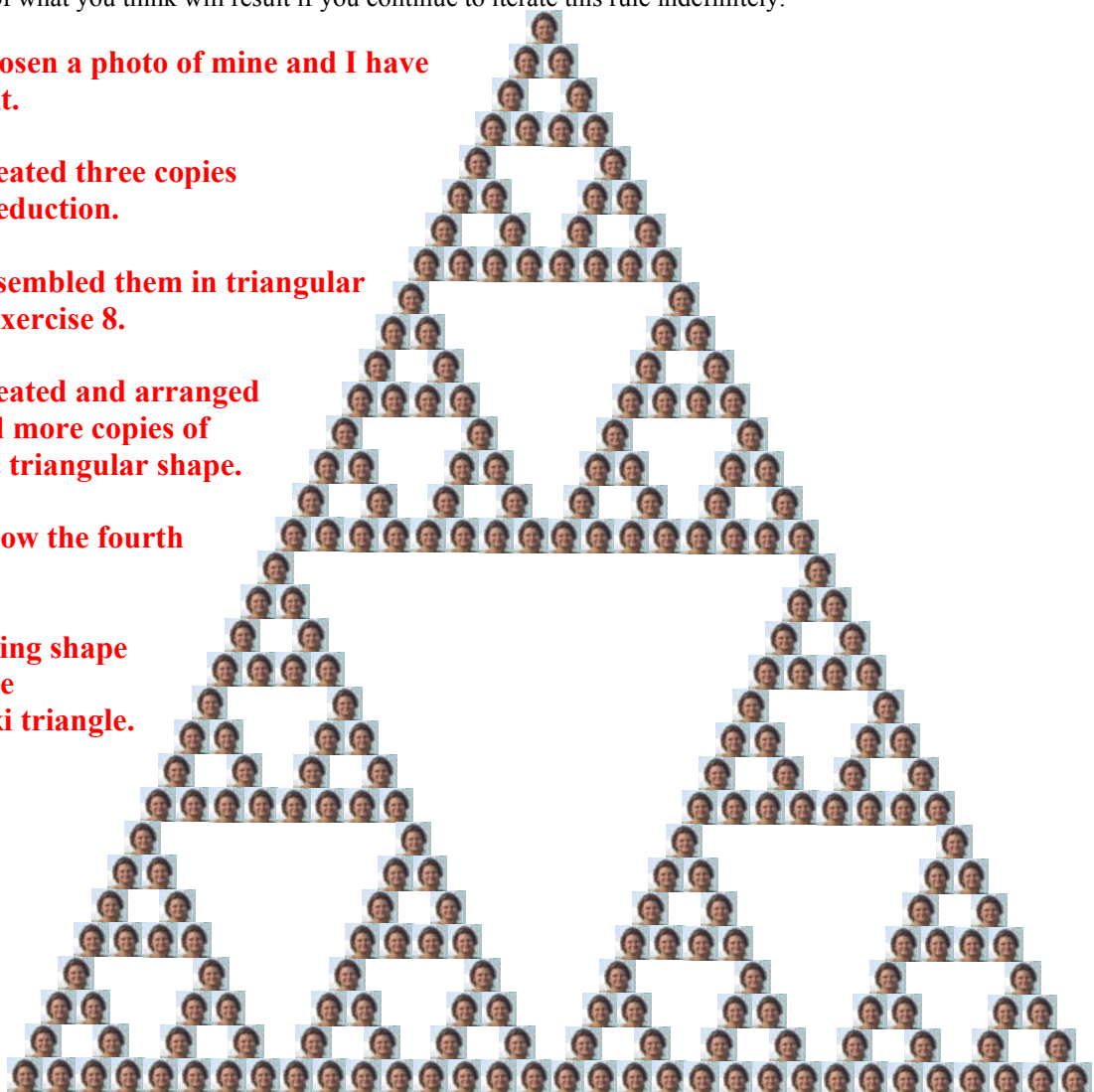
I have created three copies at 50% reduction.

I have assembled them in triangular form as exercise 8.

I have created and arranged more and more copies of this basic triangular shape.

Here I show the fourth iteration.

The limiting shape will be the Sierpiński triangle.



10-Self-similarity

Remember

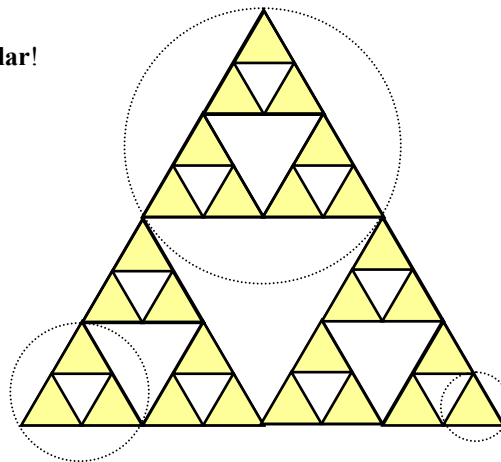
- *Similar figures are figures that have the same shape and different size.*



- *A self-similar figure is a figure made by small copies of itself.*



The Sierpiński triangle is **self-similar!**



- Take a **magnifying glass** to look more closely at the Sierpiński triangle:

Magnification is the amount by which a smaller self-similar image must be magnified to return to the original size. All magnifications are referring to linear measurements only.

1. How many copies do you see where **magnified by a factor of 2** yield the entire figure? **3**
2. And by a factor of 4? **9**
3. And by a factor of 8? **27**
4. Can you find a pattern here?

The Sierpiński triangle consists of three self-similar pieces that each have magnification factor of 2.

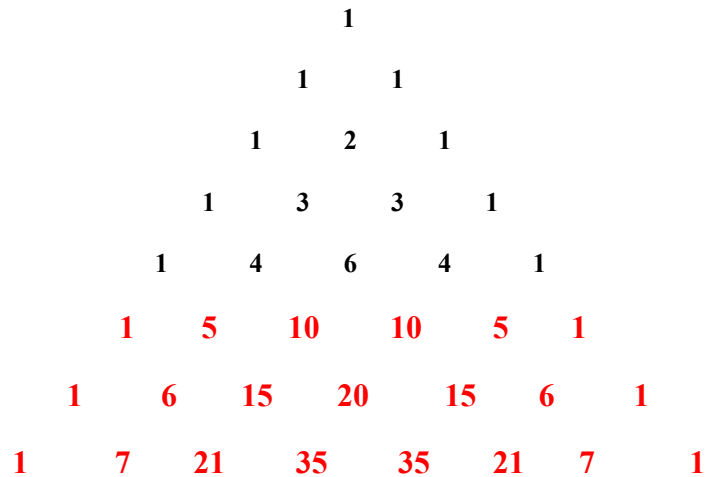
It also consists of $9 = 3^2$ self-similar pieces that have magnification factor of $4 = 2^2$.

And it also consists of $27 = 3^3$ self-similar pieces with magnification factor $8 = 2^3$.

- Go to: <http://math.rice.edu/~lanius/fractals/selfsim.html> and **zoom in**.
The triangle will look the same!

No matter how closely we look at this figure, we always find smaller and smaller copies of itself.

11-Pascal triangle



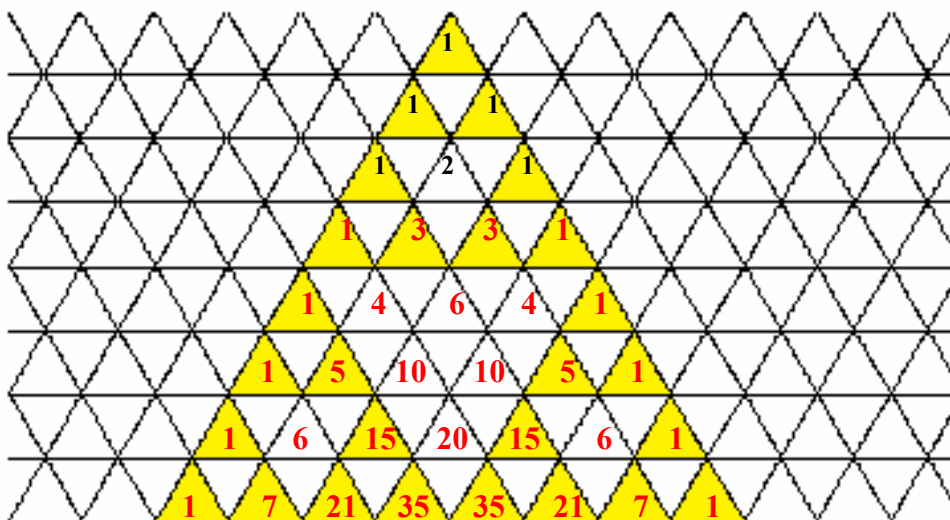
- a) Do you see the pattern of how the numbers are placed in this triangle?
Fill in the next three rows.

Any row starts and finishes by 1. We get the middle values adding the number that can be seen above and to the left and the number above and to the right. For example, in the fifth row we have $1 + 3 = 4$, $3 + 3 = 6$, $3 + 1 = 4$.

- b) This number triangle was known in China and Persia in the eleventh century. Later on the French mathematician Blaise Pascal, among others, studied its properties. In which **century** did Pascal live?

Pupils search the answer on the Internet: Seventeenth century.

- c) Copy the Pascal triangle in the grid below.
 d) **Shade** all the little **odd numbered** triangles in the Pascal triangle.



- e) What figure did you get?
The third iteration of the Sierpiński triangle fractal.

12-The chaos game

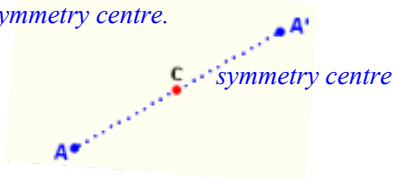


1. **Number of players:** in pairs.

2. **Materials:** worksheet, die, pencil and rubber, pen, ruler (better a half-ruler as showed on your right), transparency and marker pen.

Remember

Central symmetry is the reflection of an object through a point, called symmetry centre.



3. **Half-ruler:** It is handy to use a half-ruler for drawing. You can make your own half-ruler from cardboard.

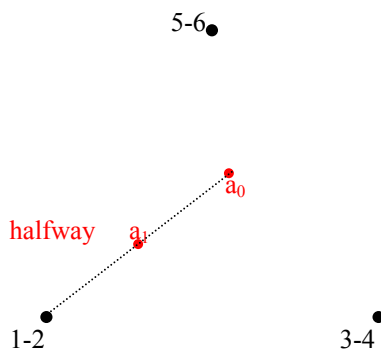
4. How to start playing?

- Place three points at the vertices of an equilateral triangle. Call them 1-2, 3-4 and 5-6.
- Choose **any** point in the interior of the triangle. Use a **pencil**. Call this point a_0 .

5. How to play?

- First **roll the die** to determine a **vertex**.
- Then, using a pencil and a half-ruler, move to a new point which is **exactly halfway** between a_0 and the target vertex. Call this point a_1 .

6. **Example:** if you roll a 2

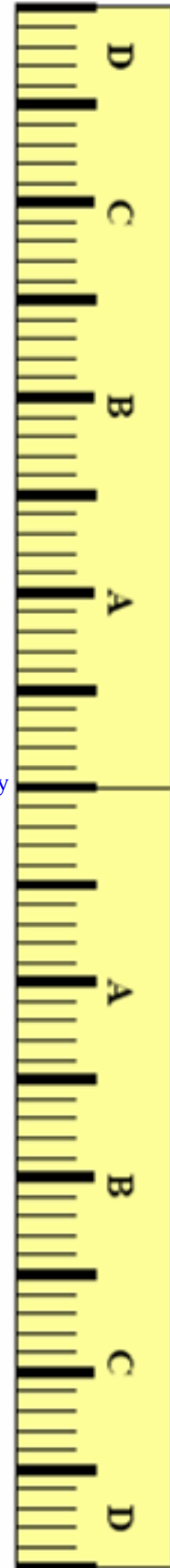


7. Playing the game

Play the chaos game with the given vertices below.

- Choose a point a_0 somewhere **inside** the triangle.
- Roll the die and plot the point a_1 using your half-ruler and **pencil**.
- Plot the points a_2, a_3, a_4, a_5 and a_6 determined by your die rolls.

halfway



Open answer.

The steps of the process will be random. In order to make sure that you truly have a random process it helps if you erase the first points in the orbit. That way, you will begin recording the orbit at a more random seed.

5-6

- Switch to a pen and plot a_7 .
- Erase the first seven points $a_0, a_1, a_2, a_3, a_4, a_5$ and a_6 .
- Plot the next ten points or more using a pen.
- Compare your results with those of your classmates.
- Copy your points onto a transparency. The rest of the class does the same.

1-2

3-4

Have two or three transparencies ready. Several pairs of pupils can record their points on the same transparency. Make sure the transparencies have the three vertices already on them to match up with worksheets.

Lining all the transparencies up, do you see a pattern emerging? **It seems that the Sierpiński triangle is emerging.**

Pupils should realise at least that the middle section of the triangle is not being filled in. The access to the Web site below will help to see the emerging shape.

8. The game on the Web

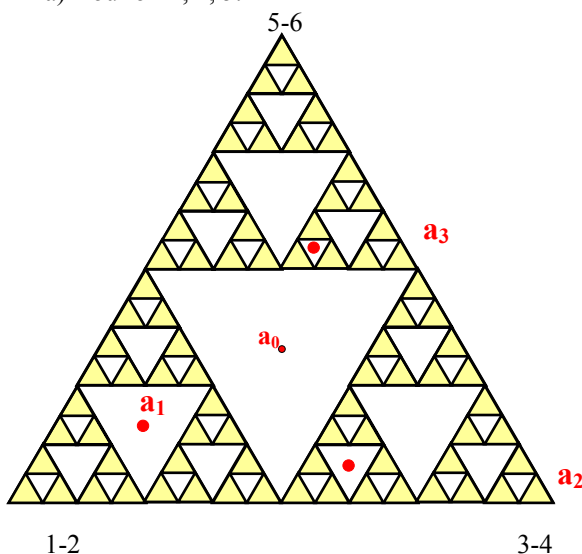
Go to <http://www.jgiesen.de/ChaosSpiel/Spiel1English.html> and play the game plotting one point at a time.

Go to <http://www.jgiesen.de/ChaosSpiel/Spiel1000English.html> and play the game plotting one thousand points at a time.

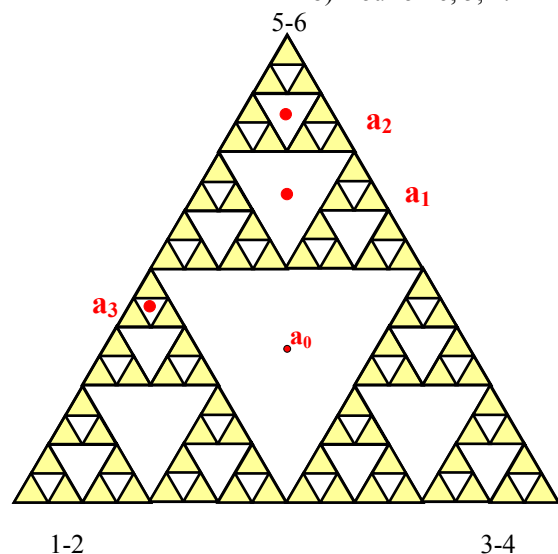
9. Playing the game with the Sierpiński triangle showing

- Choose the point a_0 in the middle of the triangle and plot the points a_1, a_2 and a_3 in each picture below according to the three rolls of the die indicated.

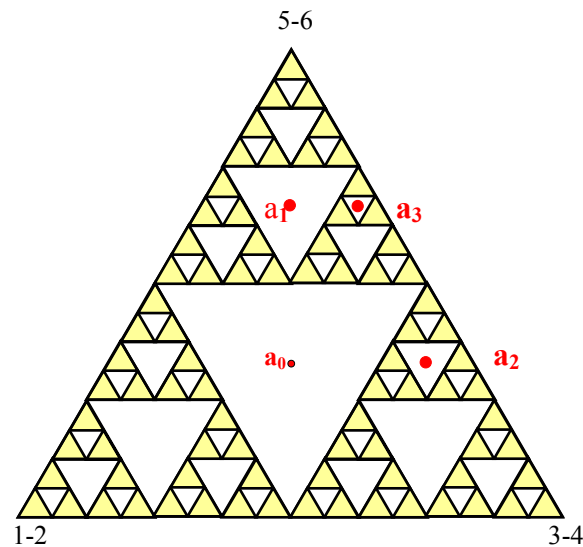
a) You roll 1, 4, 5.



b) You roll 6, 5, 2.



c) You roll 5, 3, 6.



- **Explain** how the points are distributed in the triangles.

a) The point a_1 is located halfway between a_0 and the vertex 1-2, so must land in one of the three smaller holes at the next level of the construction.

The point a_2 is located halfway between a_1 and the vertex 3-4, so must lie in one of the next smaller holes.

The point a_3 is located halfway between a_2 and the vertex 5-6, so must lie in one of the next smaller holes.

b) and c) Similar answers.

10. Conclusion

Argue with your partners and your teacher why after playing the chaos game thousands of times appears the image of the Sierpiński triangle.

In the chaos game the first points are erased so we don't see a_0 , a_1 , a_2 and a_3 in the final picture.

We see that each time we roll the die, our point moves into smaller and smaller missing triangles.

Although the point is still inside a microscopically small missing triangle, it is getting closer and closer to filled triangles. The point is slowly being attracted to the Sierpiński triangle.

Something geometrically regular can emerge from a purely random game!!!

13-The Sierpiński Tetrahedron**Go to**

http://www.toomates.net/l/istes/geometria/figures_geometriques/fractals/tetraedre_sierpinski_bastonets_gelat/tetraedre_sierpinski_bastonets_gelat.htm

Look at the pictures and **cooperate** with all of your classmates by building your own Sierpiński Tetrahedron

Materials: ice-cream sticks and glue.



- **At stage 1** you put four small tetrahedra together.

How many small tetrahedra will you need at stage 2? $4 \cdot 4 = 16$

How many small tetrahedra will you need at stage 3? $4 \cdot 4 \cdot 4 = 16 \cdot 4 = 64$

How many small tetrahedra will you need at stage 4? $4 \cdot 4 \cdot 4 \cdot 4 = 64 \cdot 4 = 256$

Can you find a pattern here? **The number of tetrahedra at any given stage is always the number of tetrahedra in the previous stage, multiplied by four.**

How many small tetrahedra would you need at stage 10? $4^{10} = 1\,048\,576$ tetrahedra

- Suppose that the length of each stick is 10 cm.

How long will each edge be at stage 1? $10 \cdot 2 = 20$ cm

How long will each edge be at stage 2? $10 \cdot 2 \cdot 2 = 40$ cm

How long will each edge be at stage 3? $10 \cdot 2 \cdot 2 \cdot 2 = 80$ cm

How long will each edge be at stage 4? $10 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 160$ cm

Can you find a pattern here? **The edge length at any given stage is always the edge length in the previous stage, multiplied by two.**

How long will each edge be at stage 10? $10 \cdot 2^{10} = 10\,240$ cm

The Sierpiński Tetrahedron is **self-similar!**

LESSON PLAN 4. Koch curve and Snowflake

Aim: To introduce pupils to one of the most popular and well known fractal. The two ways to generate fractals geometrically, by “removals” and “copies of copies”, are revisited. Pupils should begin to develop an informal concept of what fractals are.

Teaching objectives:

Content

- Geometric iteration rule by removals
- Copies of copies method
- Areas and perimeters
- Numerical and geometric patterns
- Geometric transformations. Friezes
- Self-similarity

Communication

- Predicting: “will be...”, “would be ...”,
- Comparing: “getting smaller ...”, “larger than...”, “magnified by a factor of ...”, “increasing by a factor of ...”
- Describing your work: “I have chosen ...”,...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Sharing opinions to draw a frieze
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition

- Identifying similar shapes
- Predicting eventual outcomes
- Calculating parts of a unit whole, perimeters and areas
- Creating figures through drawing programs
- Creating spreadsheets to display data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers
- Using Internet resources to confirm predictions

Context

- Zooms of the size of images
- Computer graphics and visual art
- Frieze decoration
- History of Mathematics

Outcomes: At the end of the lesson, pupils will be able to

- Distinguish between two similar figures and a self-similar figure
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Confirm predictions using the Internet resources
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Present their work to the rest of the class

Key words: geometric iteration rule, Koch curve, Koch Snowflake, self-similarity, frieze.

Prerequisites: Pupils need to be familiar with straight line segments, equilateral triangles, polygons, whole powers and fractions as a part of a whole.

Curriculum connections:

- Calculation of perimeters and areas.
- Computation with fractions in a geometric context.
- Triangle's geometry.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Use of calculator or computer when needed.

Timing: Six 55-minute class periods.

Tasks planned:

First session

- Provide pupils with worksheets 1-*Removals* and 2-*Koch curve*. Let them work on their own. Only circulate among them to be sure they are headed in the right direction.
- The indicated Web site is more interesting than others because the quick increasing of the length of the curve can easily be observed. Point out this fact.
- Plenary: sharing and discussing solutions.
- Homework: Ask to finish accurately the exercise.

Second session

- Pupils work through 3-*Length of the Koch curve*. The first two columns of the grid are known from the previous worksheet.
- Plenary: correction of exercise 3 and comments on the conclusion.

Third session

- Pupils work through exercise 4-*Copies of copies* by using a computer drawing program.
- The entire class cooperates by making ornamental friezes for decoration.
- Homework: Ask pupils to explore different possibilities.

Fourth session

- Each pupil presents to the rest of the class his/her home exploration.
- Final conclusion: teacher could take this opportunity to explain that there are only seven possible frieze patterns.
Go to: <http://mathcentral.uregina.ca/RR/database/RR.09.01/mcdonald1/>
- Homework: exercise 6-*Self-similarity*.

Fifth session

- Homework revision: Ask questions to be sure that pupils understand what self-similarity is.
- Pupils work through exercise 7-*The Koch Snowflake* and 8-*Perimeter of the Koch Snowflake*.
- Homework: Ask pupils to finish exercise 8.

Sixth session

- Homework revision.
- Pupils work through exercise 8-*Area of the Koch Snowflake*.
- Plenary: Final conclusions about perimeter and area of the Koch Snowflake.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For drawings and making friezes: ruler, pencil and rubber, coloured pencils, cardboard, scissors and glue, pen, markers ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Friezes will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Answers to exercises

1-Removals

Here is a geometric iteration:

Seed: a straight line segment

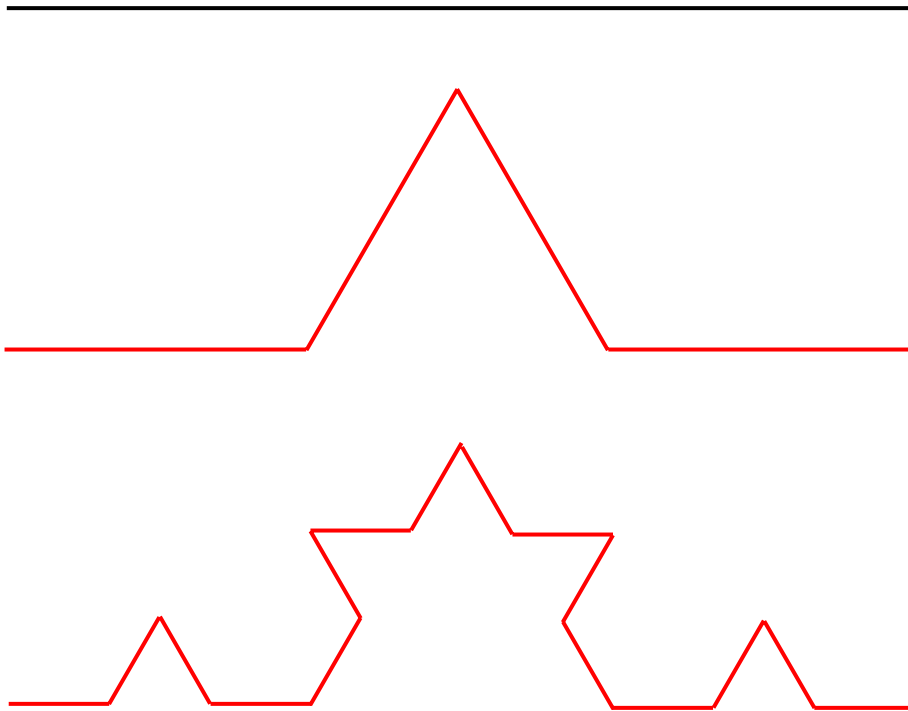


Iteration rule:

- Divide the segment into three equal parts
- **Remove** the middle third
- Replace it with two segments the same length as the section you removed.



1. Start with the straight line segment below and sketch the first and the second iteration.



2. - How many segments does the first iteration have? **4 segments**
- How many segments does the second iteration have? **4 times 4 = 4² = 16 segments**
- How many segments **will** the third iteration have? **4 times 16 = 4³ = 64 segments**
- Do you see a **pattern** here? Explain.
From every segment we get 4 smaller segments in the next iteration. So we have to multiply by 4 at each stage.

2-Koch curve

1. Go to: <http://www.arcytech.org/java/fractals/koch.shtml>

This applet is implemented in such a way that you can see the fractal as it gets drawn.

- a) Play the game and move from one iteration to the next.
Stop at the 11th iteration (it takes a long time to draw and a good memory is needed).
- b) How many segments does the eleventh iteration have?

We begin with 1 segment and we have to multiply eleven times by 4, that is $4^{11} = 4\,194\,304$ segments.

Pupils should notice that it takes a long time to draw more than 4 million small segments.

- c) In your imagination, you could repeat this rule infinitely many times.
The orbit of this iteration rule tends to famous fractal known as

the Koch curve



2. Search for information on the Web and complete the following passage by filling in the blanks

Niels Fabian Helge Koch (1870 - 1924) was born in **Stockholm**, the capital of **Sweden**.
He described this curve in the year **1904**.

3. Assume the **length** of the original straight line segment is **1 unit**.

-What **fraction** is the length of each segment at the first iteration?

1 / 3

-What fraction is the length of each segment at the second iteration?

1/3 of 1/3 = 1/3 times 1/3 = 1/9

-What fraction **will be** the length of each segment at the third iteration?

1/3 of 1/9 = 1/3 times 1/9 = 1/27

- Do you see a **pattern** here? Use this pattern to predict what fraction **would be** the length of each segment at the fourth iteration.

$$(1/3)^2 = 1/9$$

$$(1/3)^3 = 1/27$$

$$\underline{(1/3)^4 = 1/81}$$

3-Length of the Koch curve

Assume the **length** of the original straight line segment is **1 unit**.



1. Complete the grid below (round to two decimal places):

	Number of segments	Length of each segment	Total length (as a fraction)	Total length (as a decimal)
Original segment	1	1	1	1
First iteration	4	$\frac{1}{3}$	$4 \cdot (1/3) = 4/3$	1.33
Second iteration	$4^2 = 16$	$(1/3)^2 = 1/9$	$4^2 \cdot (1/3)^2 = 16/9$	1.78
Third iteration	$4^3 = 64$	$(1/3)^3 = 1/27$	$4^3 \cdot (1/3)^3 = 64/27$	2.37
Fourth iteration	$4^4 = 256$	$(1/3)^4 = 1/81$	$4^4 \cdot (1/3)^4 = (4/3)^4$	3.16

2. Complete the sentences below:

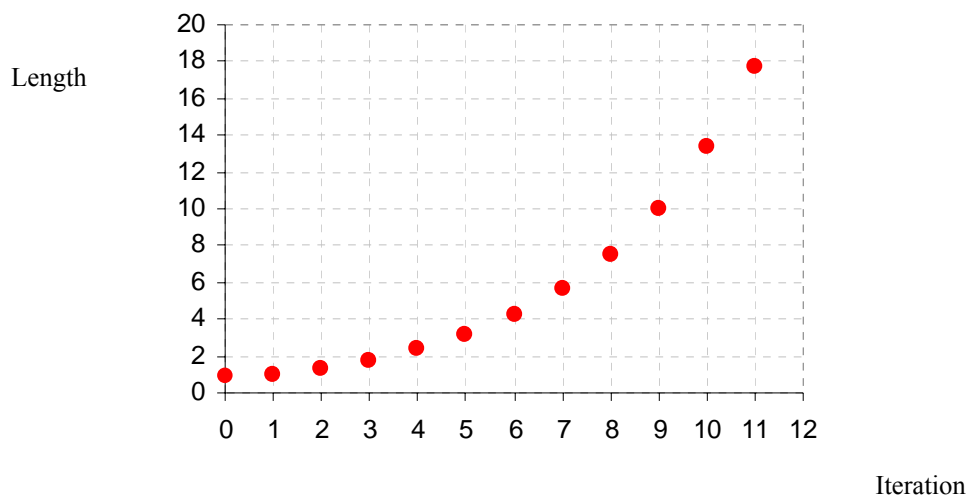
- The total length of each iteration is $4/3$ times the total length of the iteration before.
- The total length is **increasing** by a scale factor of $4/3$.
- The total length of the fifth iteration is $(4/3)^5 \approx 4.21$ units.

3. Use a spreadsheet or calculator to compute the total length of the next seven iterations.

Enter the seed 1 into the first cell A1 and the formula = A1*4/3, into the second cell A2.

4.21, 5.62, 7.49, 9.99, 13.32, 17.76 units (rounded to 2 d.p.)

4. Sketch a plot of these twelve points.



5. How many iterations would it take to obtain a total length of 100 units, or as close to 100 as you can get?

At the sixteenth iteration the length is 99.77 units (rounded to two d.p.)

6. How many iterations would you have to draw to guarantee that the length is **larger** than 200 units?

We have to draw 19 iterations, because at the eighteenth iteration the length is approximately 177.38 units and at the nineteenth iteration the length is 236.50 units.

7. Compare with your partner the answers above to number 5 and number 6. What can you explain about it?

We need 16 iterations to get a length of 100 units, but with only 3 more iterations the length is growing more than 100 units.

8. Write down a formula to calculate the length of the curve at the thousandth iteration.

$$(4 / 3)^{1\ 000}$$



9. What can you explain about the length if we continue iterating?

The length keeps increasing quite quickly. The Koch curve is infinitely long!!

4-Copies of copies



Use any computer **drawing program** in this exercise.

Start **again** with a straight line segment. **Everyone** in class should start with the **same size** segment.

The iteration rule is:

- Make four **copies** of the segment, each **reduced** to one-third the original size.



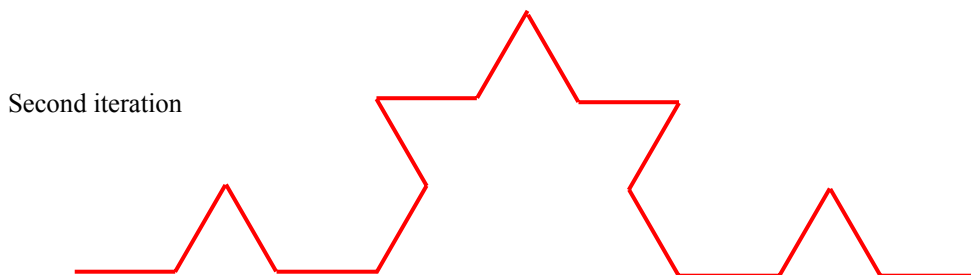
- **Assemble** in this way:

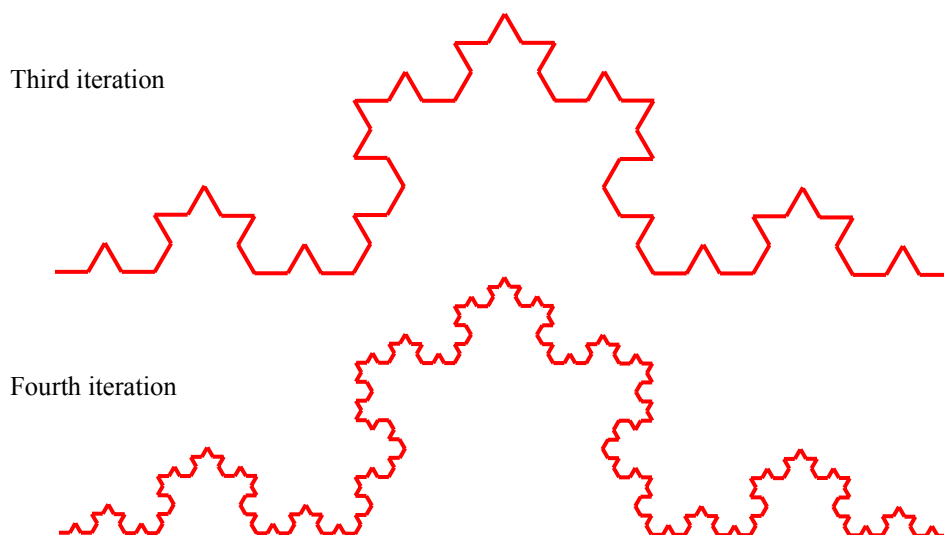


1. From left to right the second segment is **rotated counter clockwise** and the third one is rotated **clockwise**.

What is the angle of rotation? **60°**

2. Now draw the second, the third and the fourth iteration.



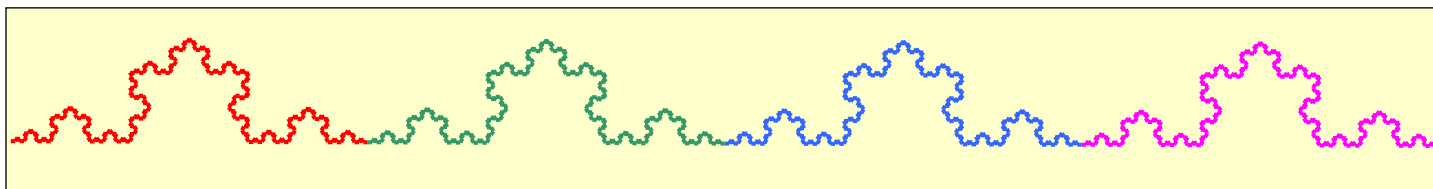


3. Go to: <http://www.xtec.cat/~dobrador/abeam/koch.swf>
 What shape will you get by continuing this process? **The Koch curve**

5-Frieze presentation

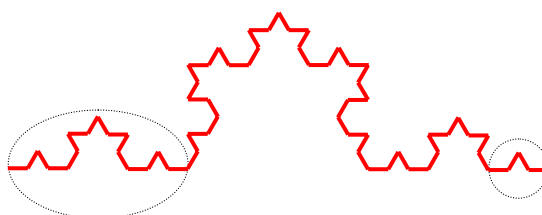
1. **Print** the fourth iteration.
2. **Cooperate** with your classmates by making an ornamental **frieze** to decorate your classroom.

Here is an example, but other possibilities could be explored.



6-Self-similarity

The Koch curve is **self-similar**!



- Take a **magnifying glass** to look more closely at the Koch curve:

Magnification is the amount by which a smaller self-similar image must be magnified to return to the original size. All magnifications are referring to linear measurements only.

- a) How many copies do you see where **magnified by a factor of 3** yield the entire figure? **4 copies**
- b) And by a factor of 9? **16 copies**

- Go to: <http://www.jimlov.com/fractals/koch.htm> and look at the movie **Zooming in on the Koch Curve**.

The curve looks the same! **Miniature copies of itself can be found all along the curve!**

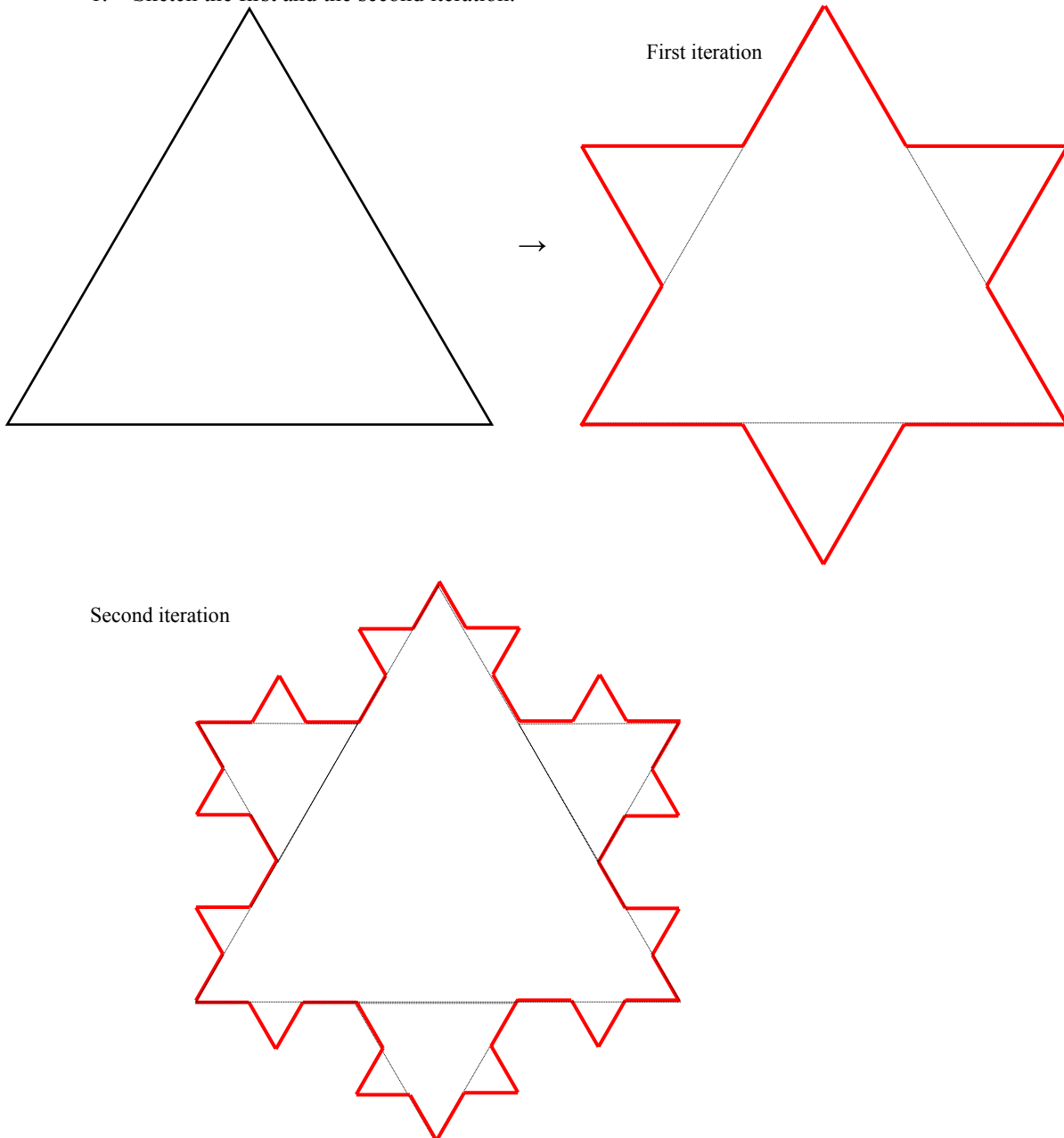
7-The Koch Snowflake

Start with an equilateral triangle.

Apply the iteration rule:

- Divide **each side** of the triangle into three equal parts.
- **Remove** the middle third.
- Replace it with two segments the same length as the section you removed.

1. Sketch the first and the second iteration.



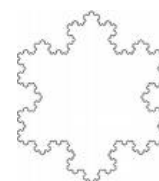
- 2. - How many sides does the first **polygon** have? **4 times 3 = 12 sides**
- How many sides does the second polygon have? **4 times 12 = 48 sides**
- How many sides **will** the third iteration have? **4 times 48 = 192 sides**
- Do you see a **pattern** here? Explain.

From every side we get 4 smaller sides in the next iteration. So we have to multiply by 4 at each stage.

3. Go to: <http://math.rice.edu/~lanius/frac/koch/koch.html>

- a) Play the game and move from one iteration to the next.
- b) How many sides does the sixth iteration have? **$4^6 \cdot 3 = 12\,288$ sides**
- c) In your imagination, you could repeat this rule infinitely many times. The orbit of this iteration rule tends to famous fractal known as

the Koch Snowflake curve



8-Perimeter of the Koch Snowflake

Assume the **perimeter** of the original triangle is **3 units**.



1. Complete the grid below (round to two decimal places):

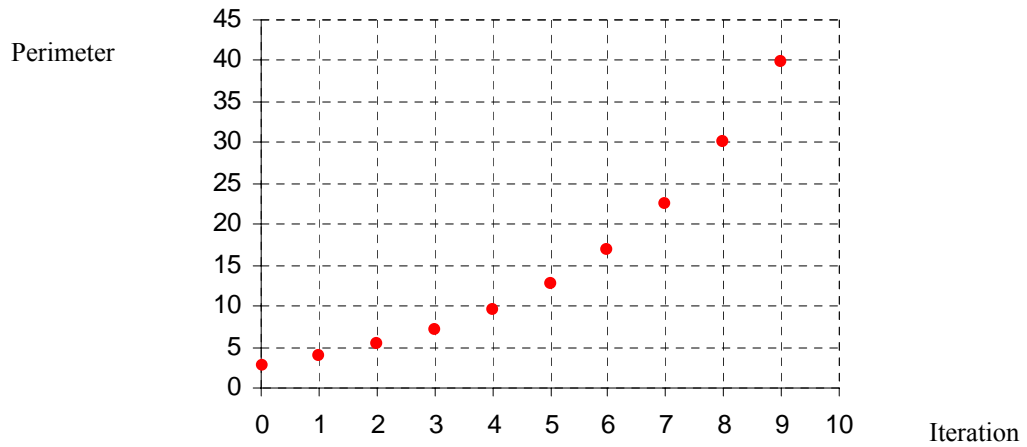
	Number of sides	Length of each side	Perimeter (as a fraction)	Perimeter (as a decimal)
Original triangle	3	1	$3 \cdot 1 = 3$	3
First iteration	$4 \cdot 3 = 12$	$\frac{1}{3}$	$12 \cdot (1/3) = 12/3 = 4$	4
Second iteration	$4^2 \cdot 3 = 48$	$(1/3)^2 = 1/9$	$4^2 \cdot 3 \cdot (1/3)^2 = 4^2 / 3 = 16/3$	5.33
Third iteration	$4^3 \cdot 3 = 192$	$(1/3)^3 = 1/27$	$4^3 \cdot 3 \cdot (1/3)^3 = 4^3 / 3^2 = 64/9$	7.11
Fourth iteration	$4^4 \cdot 3 = 768$	$(1/3)^4 = 1/81$	$4^4 \cdot 3 \cdot (1/3)^4 = 4^4 / 3^3 = 256/27$	9.48

2. Complete the sentences below:

- The perimeter of each figure is **$4/3$** times the perimeter of the figure before.
- The perimeter is **increasing** by a scale factor of **$4/3$**
- The perimeter of the fifth iteration is **$4^5 / 3^4 \approx 12.64$** units.

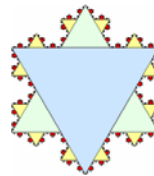
3. Use a spreadsheet or calculator to compute the perimeter of the next five iterations.
12.64, 16.86, 22.47, 29.97, 39.95 units (rounded to 2 d.p.)

4. Sketch a plot of these ten points.



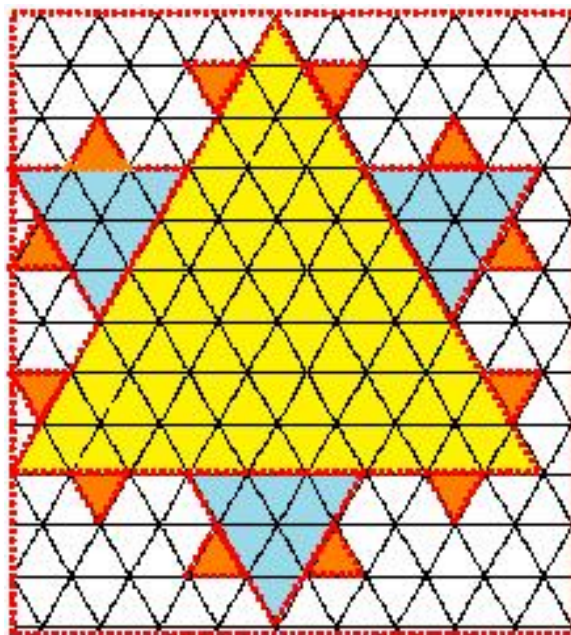
5. What can you explain about the perimeter if we continue iterating?

The perimeter increases by a factor greater than 1, so the Koch Snowflake is infinitely long!




8-Area of the Koch Snowflake

1. Use coloured pencils to sketch below how the second iteration looks like.



2. Let's measure the area using the small triangles of our grid as units.

1 area unit = 

	Number of triangles added	Area of each triangle added	Amount of area added	Total area
Original triangle				81
First iteration	3	9	27	108
Second iteration	12	1	12	120

3. Complete the sentences below:

- The number of triangles added is **4** times the previous one.
- The area of each triangle added is **1/9** the previous one.
- The amount of area added is **increasing** by a scale factor of **4/9**

4. Predict **with your partner** the next five rows of the table:

Third iteration	$3 \cdot 4^2 = 48$	$1/9$	5.33	125.33
Fourth iteration	$3 \cdot 4^3 = 192$	$1/9^2$	2.37	127.7
Fifth iteration	$3 \cdot 4^4 = 768$	$1/9^3$	1.05	128.75
Sixth iteration	$3 \cdot 4^5 = 3\ 072$	$1/9^4$	0.47	129.22
Seventh iteration	$3 \cdot 4^6 = 12\ 288$	$1/9^5$	0.21	129.43

5. What is remarkable about the area and the perimeter of this shape?

The area is increasing but more and more slowly. It tends to some number because is a finite area.

The Koch snowflake has infinite perimeter but encloses finite area!

In fact, the Koch snowflake has $8/5$ of the area of the original triangle, what are 129.6 units.

6. Go to: <http://math.rice.edu/~lanius/frac/koch3.html> and check your answers.

LESSON PLAN 5. Fractal Dimension

Aim: To revisit the traditional concept of dimension and to introduce pupils to a new type of dimension called fractal dimension. Pupils should gain a feeling for fractal dimension as a measure of the thickness or roughness of the fractal.

Teaching objectives:

Content

- Dimension: length, width and height
- Copies of copies
- Self-similarity
- Fractal dimension
- The Ruler method
- The Box Counting method

Communication

- Predicting: “will be...”, “would be ...”
- Comparing: “magnified by a factor of ...”
- Describing your work: “I have chosen ...”,...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in pairs
- Reading summary sentences to other groups

Cognition

- Identifying length, width and height
- Identifying similar shapes
- Developing intuition for dimension
- Measuring dimensions through iterative procedures
- Predicting eventual outcomes
- Creating figures through drawing programs
- Linearising data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers

Context

- Zooms of the size of images
- Computer graphics and visual art

Outcomes: At the end of the lesson, pupils will be able to

- Measure dimensions through iterative procedures
- Identify numerical and geometric patterns
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in geometric English language

- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion

Key words: Sierpiński triangle, Koch curve, self-similarity, fractal dimension, Ruler method, Box Counting method.

Prerequisites: Pupils need to be familiar with the Sierpiński triangle and the Koch curve from the previous units and with exponents.

Curriculum connections:

- Positive and negative powers of 10.
- Computation with exponents in a geometric context.
- Dimension.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Linear functions.
- Use of calculator or computer when needed.

Timing: Four 55-minute class periods.

Tasks planned:

First session

- Provide pupils with worksheets 1-*Dimension* and 2-*The Sierpiński triangle*. Be sure that pupils are entering into the concept of dimension.
- Plenary: explanation and comments on the fractal dimension of the Sierpiński triangle.
- Homework: exercise 3-*The Sierpiński Tetrahedron*.

Second session

- Homework revision: Sharing opinions about fractional dimension.
- Pupils work through 4-*The Koch curve*.
- Plenary: correction of exercise 4 and comments on the conclusion.

Third session

- Pupils work through exercise 5- *Measuring the Koch curve*. Let them experiment with the log key in their calculators without knowing logarithms. Maybe negative exponents would need an explanation. You can take this opportunity to talk about direct proportion and slope of a straight line.
- Plenary: correction of exercise 5 and comments on the conclusion.

Fourth session

- Pupils work through exercise 6- *The Box Counting method*.
- Plenary: Sharing results and final conclusions about the fractal dimension of the Koch curve.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For *The Ruler and the Box Counting methods*: a strip of paper, ruler and compasses, pencil and rubber, coloured pencils, scissors, pen, markers ...
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Answers to exercises

1-Dimension

Complete the sentences with the words in the box.

one	two	three	length	width	depth
-----	-----	-------	--------	-------	-------

1. A straight line segment has **one** dimension: **length**



2. The interior of a square has **two** dimensions: **length** and **width**



3. The interior of a cube has **three** dimensions: **length, width** and **depth**



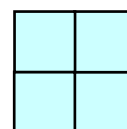
2-The Sierpiński triangle

1. Take the straight line segment below and double its length.



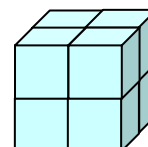
How many copies of the original segment do you get? **Two copies**

2. Take the square and double its length and width



How many copies of the original square do you get? **Four copies**

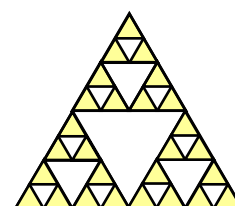
3. Take the cube and double its length, width and height.



How many copies of the original cube do you get? **Eight copies**

4. Take the Sierpiński triangle and double the length of his sides

How many copies of the original triangle do you get? **Three copies**



5. Organise your information into the table

Figure	Dimension	Number of copies	Number of copies as a power of two
straight line segment	1	2	2^1
square	2	4	2^2
cube	3	8	2^3
Sierpiński triangle	?	3	$2^?$

Do you see a pattern here? **When we double the sides and get a self-similar figure, we write the number of copies as a power of 2 and the exponent is the dimension.**

Experiment with the **exponent** key of your calculator to find out the fractal dimension of the Sierpiński triangle (**correct** to two decimal places).

The fractal dimension of the Sierpiński triangle will be the exponent d such that $2^d = 3$.

As $2^1 = 2$ and $2^2 = 4$, the dimension d will be a number between 1 and 2.

Let's try

$2^{1.5} \approx 2.828427124$ $2^{1.6} \approx 3.031433133$ $2^{1.57} \approx 2.969047141$ $2^{1.58} \approx 2.989698497$

$2^{1.59} \approx 3.010493494$

The fractal dimension of the Sierpiński triangle will be approximately 1.58, correct to two decimal places.

3-The Sierpiński Tetrahedron

- Take the Sierpiński tetrahedron and double the edge length

- How many copies of the original tetrahedron do you get? **Four copies**



- Write down the number of copies as a power of two. **2^2**

- What is the fractal dimension of the Sierpiński tetrahedron? **2, a whole number!**

Fractals don't need to have fractional dimension. Indeed, a fractal object must have fractal dimension greater than his topological dimension.

4- The Koch curve

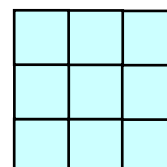
1. Take the straight line segment below and triple its length.



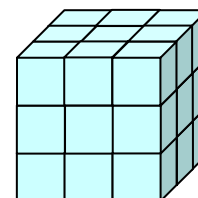
How many copies of the original segment do you get? **Three copies**

2. Take the square and triple its length and width

How many copies of the original square do you get? **Nine copies**

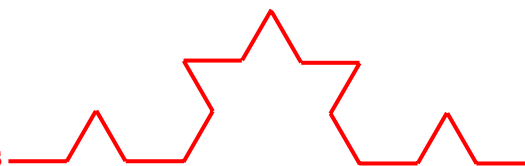


3. Take the cube and triple its length, width and height.



How many copies of the original cube do you get? **Twenty seven copies**

4. Take the Koch curve and magnify it by a factor of 3



How many copies of the original curve do you get? **Four copies**

5. Organise your information into the table

Figure	Dimension	Number of copies	Number of copies as a power of three
straight line segment	1	3	3^1
square	2	9	3^2
cube	3	27	3^3
Koch curve	?	4	$3^?$

Do you see a pattern here? **When we magnify by a factor of 3 and get a self-similar figure, we write the number of copies as a power of 3 and the exponent is the dimension.**

Experiment with the **exponent key** of your calculator to find out the fractal dimension of the Koch curve (**correct** to two decimal places).

The fractal dimension of the Koch curve will be the exponent d such that $3^d = 4$.

As $3^1 = 3$ and $3^2 = 9$, the dimension d will be a number between 1 and 2.

Let's try

$3^{1.2} \approx 3.737192818$ $3^{1.3} \approx 4.171167510$ $3^{1.25} \approx 3.948222038$ $3^{1.26} \approx 3.991836831$

$3^{1.27} \approx 4.035933423$

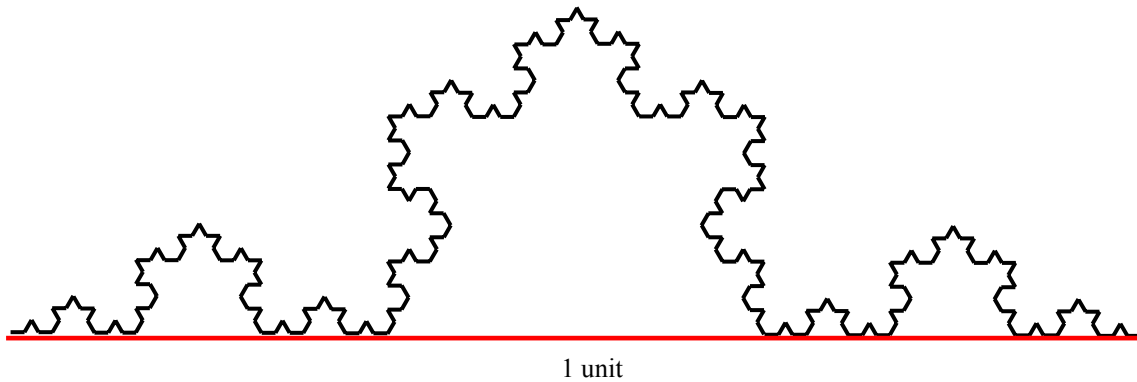
The fractal dimension of the Koch curve will be approximately 1.26, correct to two decimal places.

6. Compare the fractal dimension of the Koch curve to the fractal dimension of the Sierpiński triangle.

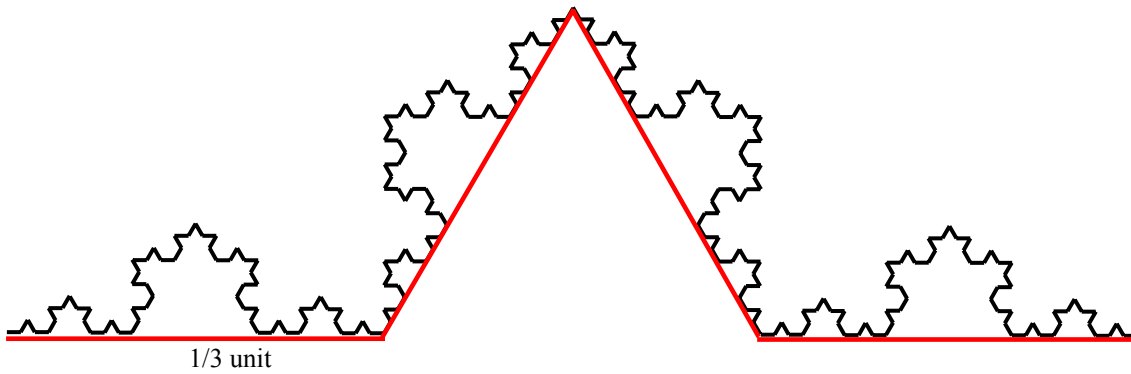
The fractal dimension of the Koch curve is smaller than the fractal dimension of the Sierpiński triangle. This is just as we could expect, as the Koch curve appears to be less thick than the Sierpiński triangle.

5- Measuring the Koch curve

Materials: Make your own ruler by cutting a strip of paper 15 cm = 1 unit long. A compass would be useful too.

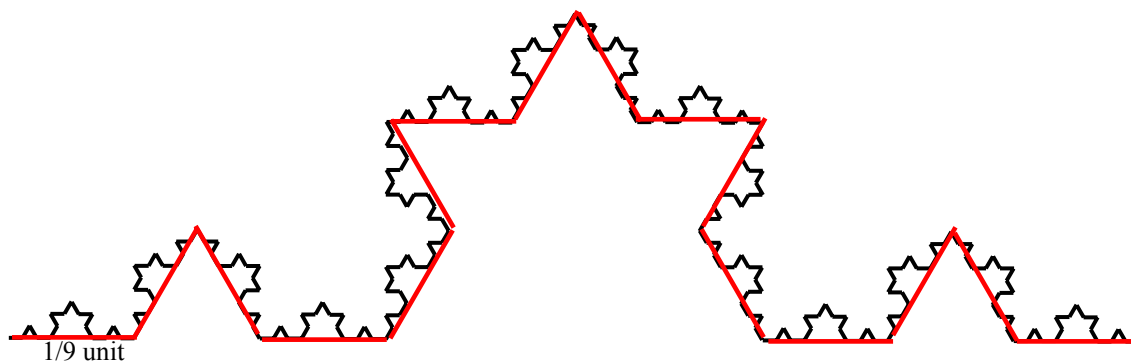


1. Folding your previous ruler into three parts measure the Koch curve.



How many $1/3$ unit long rulers do you need? **4 rulers**

2. How many $1/9$ unit long rulers do you need to measure the Koch curve? **16 rulers**



3. Do you see a pattern here?

If we use a ruler whose length is $1/3^2$, then we must use exactly 4^2 of these rulers.

4. Complete the grid below:

	Ruler length	Num of rulers
Stage 0	1	1
Stage 1	$\frac{1}{3} = 0.\overline{3}$	4
Stage 2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9} = 0.\overline{1}$	$4^2 = 16$
Stage 3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27} = \mathbf{0.0370\dots}$	$4^3 = 64$
Stage 4	$\left(\frac{1}{3}\right)^4 = \frac{1}{81} = \mathbf{0.0123456790\dots}$	$4^4 = 256$

The smaller the ruler, the more rulers we need to measure the length of the Koch curve.

Notice that in the next questions we linearise data avoiding logarithms. Pupils can experiment with the log key in their calculators without knowing this specific concept.

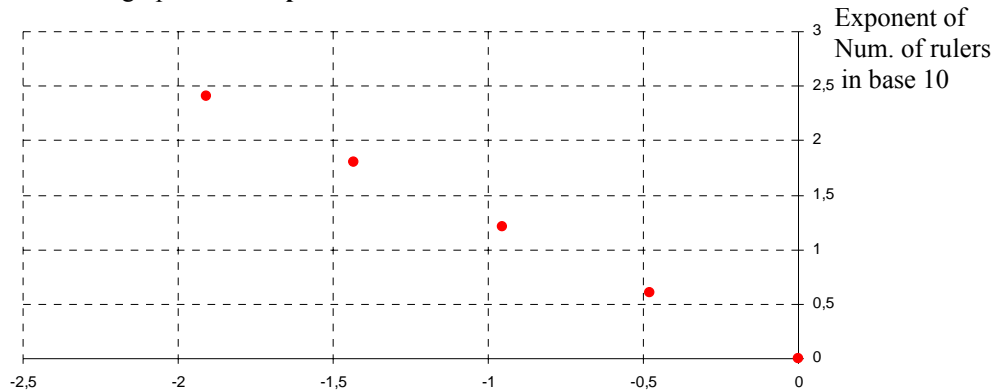
5. Experiment with the **exponent key 10^x** of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

	Ruler length	Num of rulers
Stage 0	10^0	10^0
Stage 1	$10^{-0.47}$	$10^{0.60}$
Stage 2	$10^{-0.95}$	$10^{1.20}$
Stage 3	$10^{-1.43}$	$10^{1.80}$
Stage 4	$10^{-1.90}$	$10^{2.40}$

6. Complete the table:

Exponent of ruler length in base 10	0	-0.47	-0.95	-1.43	-1.90
Exponent of Num of rulers in base 10	0	0.60	1.20	1.80	2.40

7. Sketch the graph of the exponents in the table above:



Exponent of ruler length in base 10

8. - Do these points lie on a straight line? **It would appear so.**

- Calculate the ratio $\frac{\text{Exponent of Num. of rulers in base 10}}{\text{Exponent of ruler length in base 10}}$

$$\frac{0.60}{-0.47} \approx -1.27 \quad \frac{1.20}{-0.95} \approx -1.26 \quad \frac{1.80}{-1.43} \approx -1.25 \quad \frac{2.40}{-1.90} \approx -1.26$$

Actual plotted points do not all lie on a straight line, but they are very close to one.

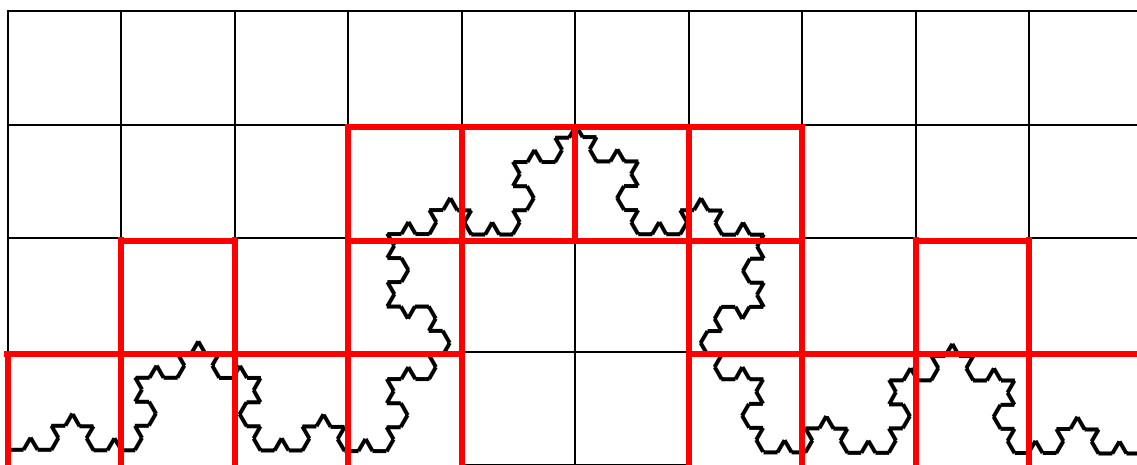
You can take this opportunity to talk about direct proportion and slope of a straight line.

- Do you recognise these values?

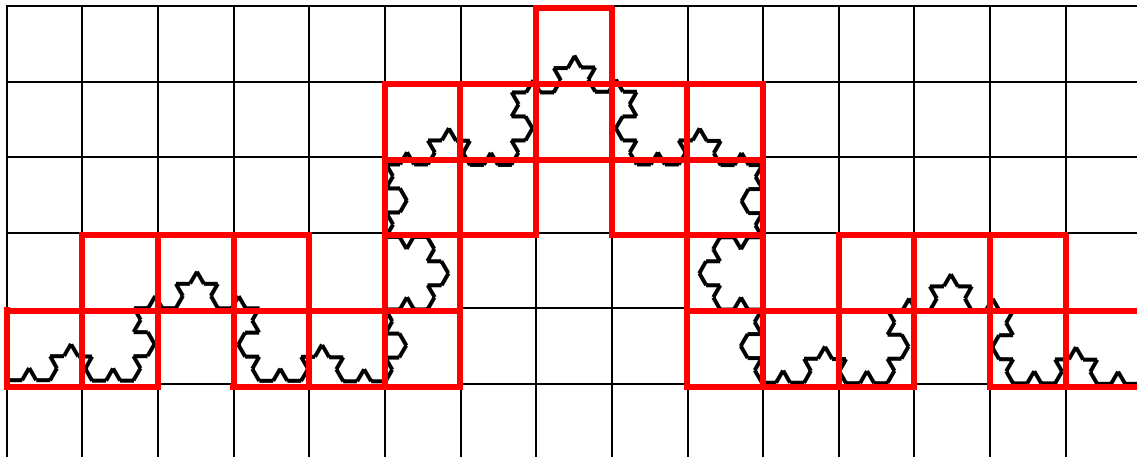
Sign changing we get an approximation of the fractal dimension of the Koch curve!

6- The Box Counting method

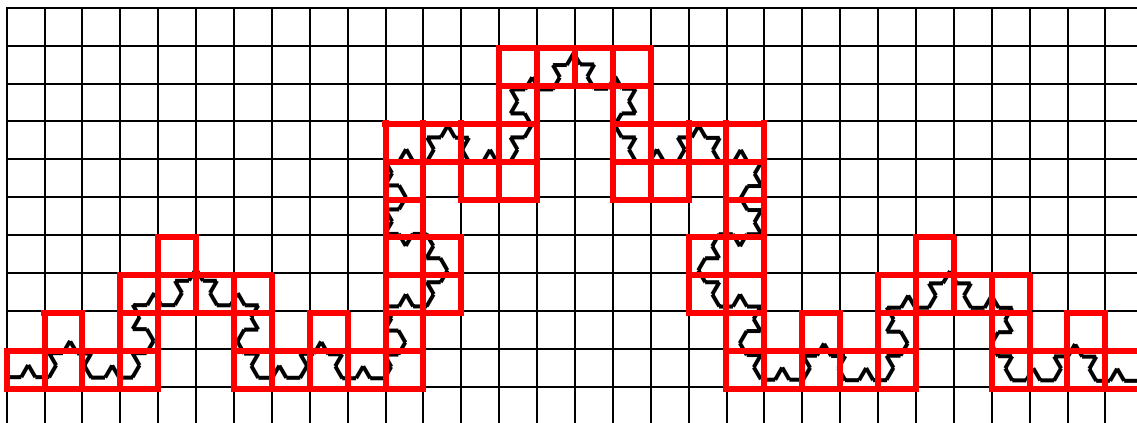
1. Paint the boxes that contain some of the Koch curve:



10-sized grid



15-sized grid



30-sized grid

2. Count the number of painted boxes and fill in the table below:

Grid size	10	15	30
Num. of boxes needed to cover the Koch curve	16	26	68

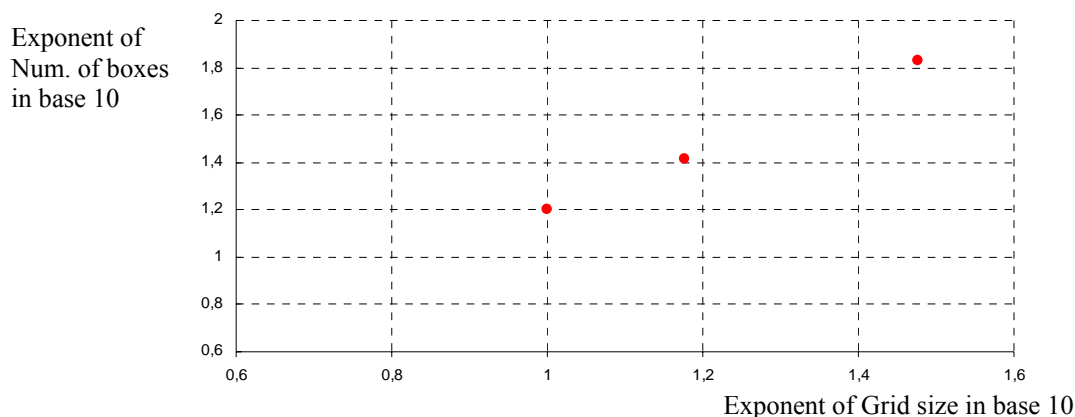
3. Experiment with the exponent key 10^x of your calculator to rewrite in base 10 the entries in the table above (correct to two decimal places):

Grid size	10^1	$10^{1.17}$	$10^{1.47}$
Num. of boxes needed to cover the Koch curve	$10^{1.20}$	$10^{1.41}$	$10^{1.83}$

4. Complete the table:

Exponent of Grid size in base 10	1	1.17	1.47
Exponent of Num. of boxes in base 10	1.20	1.41	1.83

5. Sketch the graph of the exponents in the table above:



6. - Do these points lie on a straight line? **It would appear so.**

- Calculate the ratio $\frac{\text{Exponent of Num. of boxes in base 10}}{\text{Exponent of Grid size in base 10}}$

$$\frac{1.20}{1} \approx \mathbf{1.20}$$

$$\frac{1.41}{1.17} \approx \mathbf{1.20}$$

$$\frac{1.83}{1.47} \approx \mathbf{1.24}$$

- Do you recognise these values?

We get an approximation of the fractal dimension of the Koch curve!

LESSON PLAN 6. Natural Fractals

Aim: To explore hands on fractal dimension of some natural objects. Fractals arise in many more areas than just mathematics and can be used as mathematical models of many of the complicated shapes that appear in nature.

Teaching objectives:

Content

- Geometric iteration rule
- Geometric transformations. Rotation.
- Copies of copies
- Self-similarity
- Fractal dimension
- The Ruler method
- The Box Counting method

Communication

- Predicting: “will be...”, “would be ...”
- Comparing: “magnified by a factor of ...”
- Describing your work: “I have chosen ...”,...
- Arguing with partners predictions and conclusions
- Translating visual information into verbal descriptions
- Checking calculations in cooperative groups
- Sharing opinions to organise a poster
- Reading summary sentences to other groups
- Writing a script for an oral presentation

Cognition

- Identifying iteration rules
- Identifying similar shapes
- Developing intuition for fractal dimension
- Estimating dimensions through iterative procedures
- Predicting eventual outcomes
- Creating figures through drawing programs
- Linearising data
- Plotting a graph to illustrate selected information
- Writing summary reports using key vocabulary
- Arguing and checking answers

Context

- Images development
- Zooms of the size of images
- Computer graphics and visual art
- Modelling natural fractals

Outcomes: At the end of the lesson, pupils will be able to

- Estimate dimensions through iterative procedures
- Identify numerical and geometric patterns in different contexts
- Translate visual information to verbal descriptions
- Use drawing programs to generate further iterative outputs
- Predict what will happen if we continue iterating
- Be confident in geometric English language
- Develop a sense of responsibility when working in a cooperative group
- Recognise the importance of arguing in a team
- Know how to behave in a discussion
- Know how to make a script for an oral presentation
- Present their work to the rest of the class

Key words: Self-similarity, fractal dimension, Ruler method, Box Counting method.

Prerequisites: Pupils need to be familiar with self-similarity, fractal dimension, Ruler method, Box Counting method from the previous units.

Curriculum connections:

- Positive and negative powers of 10.
- Computation with exponents in a geometric context.
- Dimension.
- Practise with recognizing and describing numerical and geometric patterns.
- Geometric transformations. Similarity.
- Cartesian coordinates and graph plotting.
- Linear functions.
- Use of calculator or computer when needed.

Timing: Six 55-minute class periods.

Tasks planned:

First session

- Pupils work through 1-*Plant-growth*.
- Plenary: comments on self-similarity and fractal dimension.
- Homework: explore another plant growing.

Second session

- Homework revision: Each pupil presents to the rest of the class his/her home project.
- Ask pupils to start working with exercise 2-*Leaf outline* and to finish it as homework.

Third session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated.
You can take this opportunity to revisit direct proportion and slope of a straight line.
- Cooperative groups start making a poster following the instructions from exercise 3-*Further exploration*.

Fourth session

- Plenary: poster presentation and oral process explanation.
- Pupils work through exercise 4- *Cauliflower* in cooperative groups. This exercise reinforces self-similarity.
- Homework: Finish the remaining calculations.

Fifth session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated.
- Provide pupils with worksheet 5-*Coastline*. Cooperative groups work accurately through this exercise. Compasses would be useful.
- Homework: Ask pupils complete the remaining questions.

Sixth session

- Homework revision: Sharing results and final conclusions about the fractal dimension estimated. You can take this opportunity to talk about line of the best fit.
- Pupils work through exercise 6- *Bone cross-section* to test their feeling for fractal dimension.
- Plenary: Summary discussion.
- Invite your pupils to search for natural fractals in the Internet.

Resources

- For class activities and homework: the pupils' worksheets and access to a computer drawing program.
- For practising *The Ruler and the Box Counting methods*: a strip of paper, ruler and compasses, transparent square grids, pencil and rubber, coloured pencils, calculator...
- For making posters: leaves of different shapes from the park next to your school, cardboard, scissors and glue, pen, markers ...
- For estimating self-similarity and fractal dimension: cauliflower or broccoli, knife and magnifying glass.
- For teaching instructions and answer keys to each exercise: the teachers' worksheets, mainly borrowed from the book *Fractals* by J. Choate, R.L. Devaney, A. Foster, listed in the bibliography.

Assessment

- The teacher will take notes of pupils' progress every period.
- Good presentation in worksheets and active participation in the classroom is essential.
- Posters will be individually marked depending on participation, contribution to the group work and oral presentation.
- Homework will be suitably marked.
- Pupils who talk in English will be given extra marks.

Evaluation

TEACHER'S NOTES AND ANSWERS

Answers to exercises

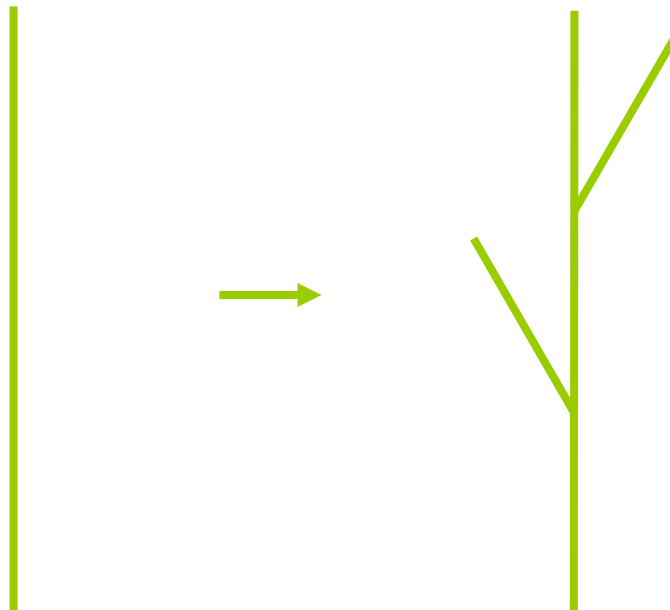
1-Plant-growth

1. Use any computer **drawing program** in this exercise.

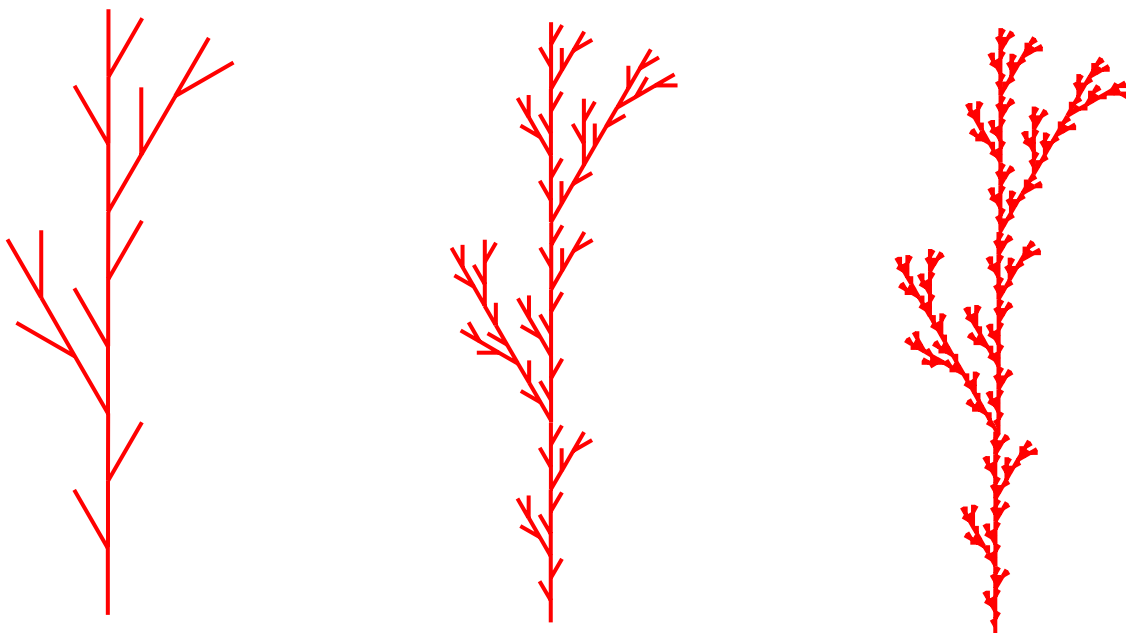
Start with a straight line segment. **Everyone** in class should start with the **same long** segment.

Apply the following iteration rule:

- Reduce the segment by one third and make five copies.
- Rotate one copy 30° clockwise and another 30° anticlockwise.
- Assemble in this way:



Now draw the second, the third and the fourth iteration.



2. If we continue iterating we get a fractal plant.
 - To investigate self-similarity and its fractal dimension, fill in the blanks:



After magnifying by a factor of 3 we get **five** copies of the original plant.

So the fractal dimension will be the exponent d such that $3^d = 5$.

- Experiment with the **exponent key** of your calculator to find out the fractal dimension (**correct** to two decimal places).

As $3^1 = 3$ and $3^2 = 9$, the dimension d will be a number between 1 and 2.

Let's try

$3^{1.3} \approx 4.171167510$ $3^{1.4} \approx 4.655536721$ $3^{1.5} \approx 5.196152422$ $3^{1.47} \approx 5.027687135$

$3^{1.46} \approx 4.972754646$

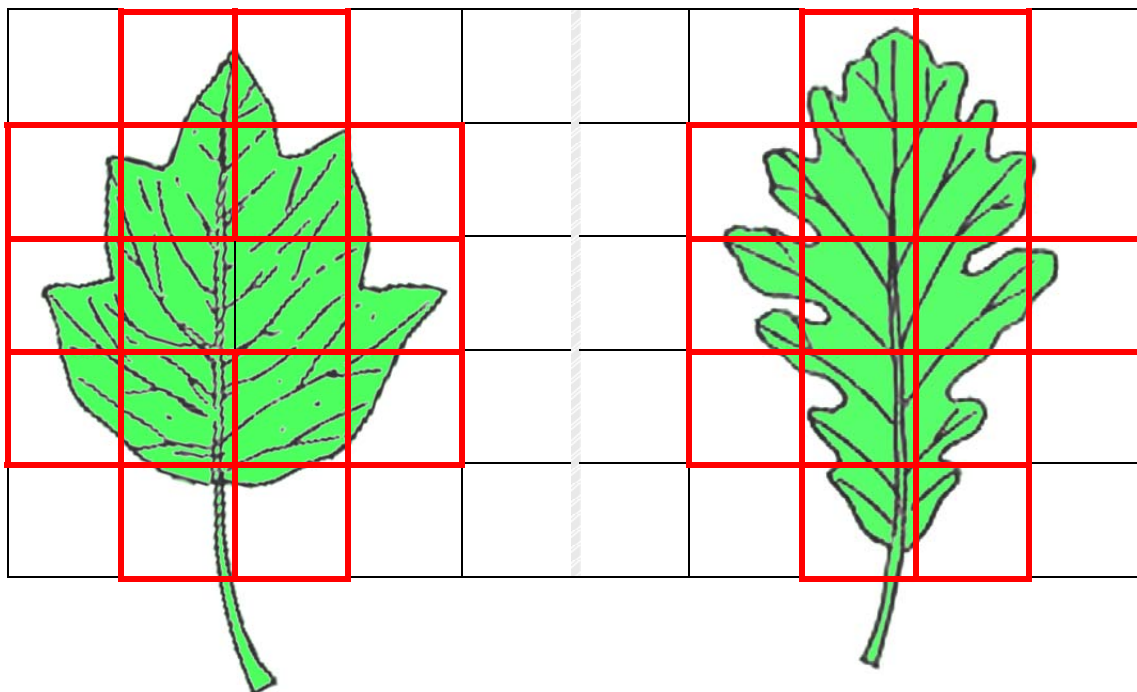
The fractal dimension will be approximately 1.46, correct to two decimal places.

- Compare to the fractal dimension of the Koch curve.

The fractal dimension of the Koch curve is smaller than the fractal dimension of this plant. This is just as we could expect, as the Koch curve appears to be less thick than the plant.

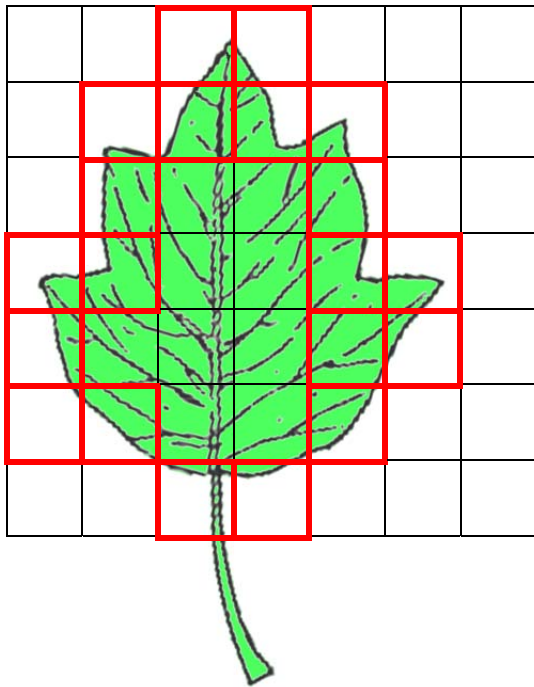
2-Leaf outline

1. Paint the boxes below that contain some of the leaf outlines:

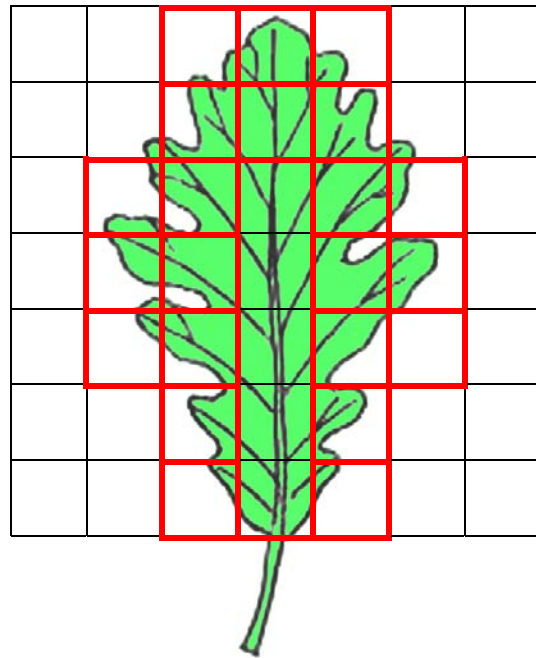


5-sized grid

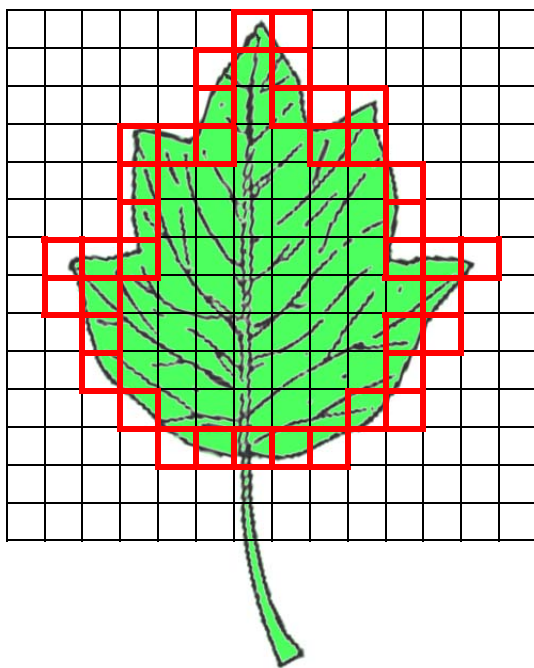
5-sized grid



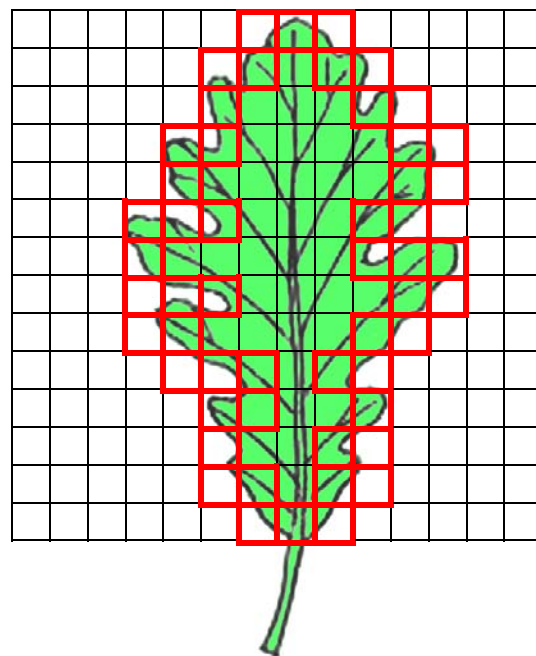
7-sized grid



7-sized grid



14-sized grid



14-sized grid

2. Count the number of painted boxes and fill in the table below:

Grid size	5	7	14
Num. of boxes covering the leaf contour 1	14	20	39
Num. of boxes covering the leaf contour 2	15	22	53

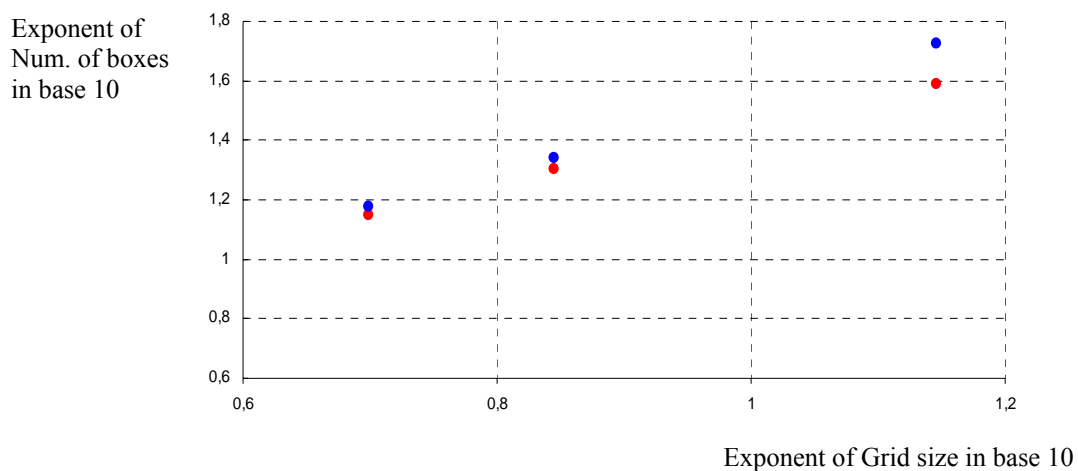
3. Experiment with the **exponent key** 10^x of your calculator to **rewrite in base 10** the entries in the table above (**correct** to two decimal places):

Grid size	$10^{0.69}$	$10^{0.84}$	$10^{1.14}$
Num. of boxes covering the leaf contour 1	$10^{1.14}$	$10^{1.30}$	$10^{1.59}$
Num. of boxes covering the leaf contour 2	$10^{1.17}$	$10^{1.34}$	$10^{1.72}$

4. Complete the table:

Exponent of Grid size in base 10	0.69	0.84	1.14
Leaf 1: Exponent of Num. of boxes in base 10	1.14	1.30	1.59
Leaf 2: Exponent of Num. of boxes in base 10	1.17	1.34	1.72

5. Sketch the graphs of the exponents in the table above (use two colours):



Legend: ■ Leaf1 ■ Leaf 2

6. Calculate the ratio $\frac{\text{Exponent of Num. of boxes in base 10}}{\text{Exponent of Grid size in base 10}}$

Leaf 1:

$$\frac{1.14}{0.69} \approx \mathbf{1.65}$$

$$\frac{1.30}{0.84} \approx \mathbf{1.54}$$

$$\frac{1.59}{1.14} \approx \mathbf{1.39}$$

Leaf 2:

$$\frac{1.17}{0.69} \approx \mathbf{1.69}$$

$$\frac{1.34}{0.84} \approx \mathbf{1.59}$$

$$\frac{1.72}{1.14} \approx \mathbf{1.50}$$

You can take this opportunity to talk about direct proportion and slope of a straight line.

7. What are the fractal dimensions of these leaf outlines?

We would say 1.5 and 1.6 approximately, but only three measures are not enough to quantify the fractal dimension.

8. What does the fractal dimension measure? Argue with your partner.

Fractal dimension estimates the complexity of leaf outline. More complex leaf outlines have higher fractal dimension. As we see in our leaves, the contour of the second leaf is more complex than the contour of the first one. Fractal dimension quantifies this complexity.

3-Further exploration

Cooperative group: Three or four classmates.

Materials:

- Leaves of different shapes from the park next to your school.
- Different sized square grids printed on transparencies.
- Calculator, cardboard, glue, pen, coloured pencils, markers...

Work: Calculate the fractal dimension of the outline of your leaves.

Poster presentation:

- Order your leaves by fractal dimension from the lowest to the highest.
- Glue them on a cardboard.
- Write down its fractal dimension.
- Present your poster to the rest of the class.

Go to: <http://hypertextbook.com/facts/2002/leaves.shtml> for more information.

4-Cauliflower

Cauliflower



Broccoli



- Cauliflower and broccoli are fractals because they branch off into smaller and smaller pieces, which are similar in shape to the original.

Cooperative group: Three or four classmates.

Materials:

- Cauliflower or broccoli.
- Knife and magnifying glass.

Aim: Explore self-similarity and determine the fractal dimension of cauliflower and broccoli.

Procedure:

- Count the number of branches growing off the main trunk.
- Cut off these branches.
- They are copies of the whole. Estimate on average the magnification factor.
- Repeat the previous steps on all of these branches. Take the mean.
- Repeat again. Eventually, you will need a magnifying glass to see the last branches you cut off.

- Complete the sentence: **Open answer**

Every branch carries aroundbranchestimes smaller.

The fractal dimension will be approximately the exponent d such that

$$(Mean\ of\ number\ of\ branches)^d = Mean\ of\ magnification\ factors$$

- Experiment with the **exponent key** of your calculator to find out the fractal dimension (**correct** to two decimal places).

Open answer between two and three.

- Compare your dimension to the dimension from other groups.

5-Coastline

Cooperative group: Three or four classmates.

Aim: Calculate the fractal dimension of *Estany de Banyoles (Catalonia)*.

Other coastlines are possible.

Materials:

- Map of the coastline
- Make your own ruler by cutting a strip of paper 16 cm = 1 unit long.

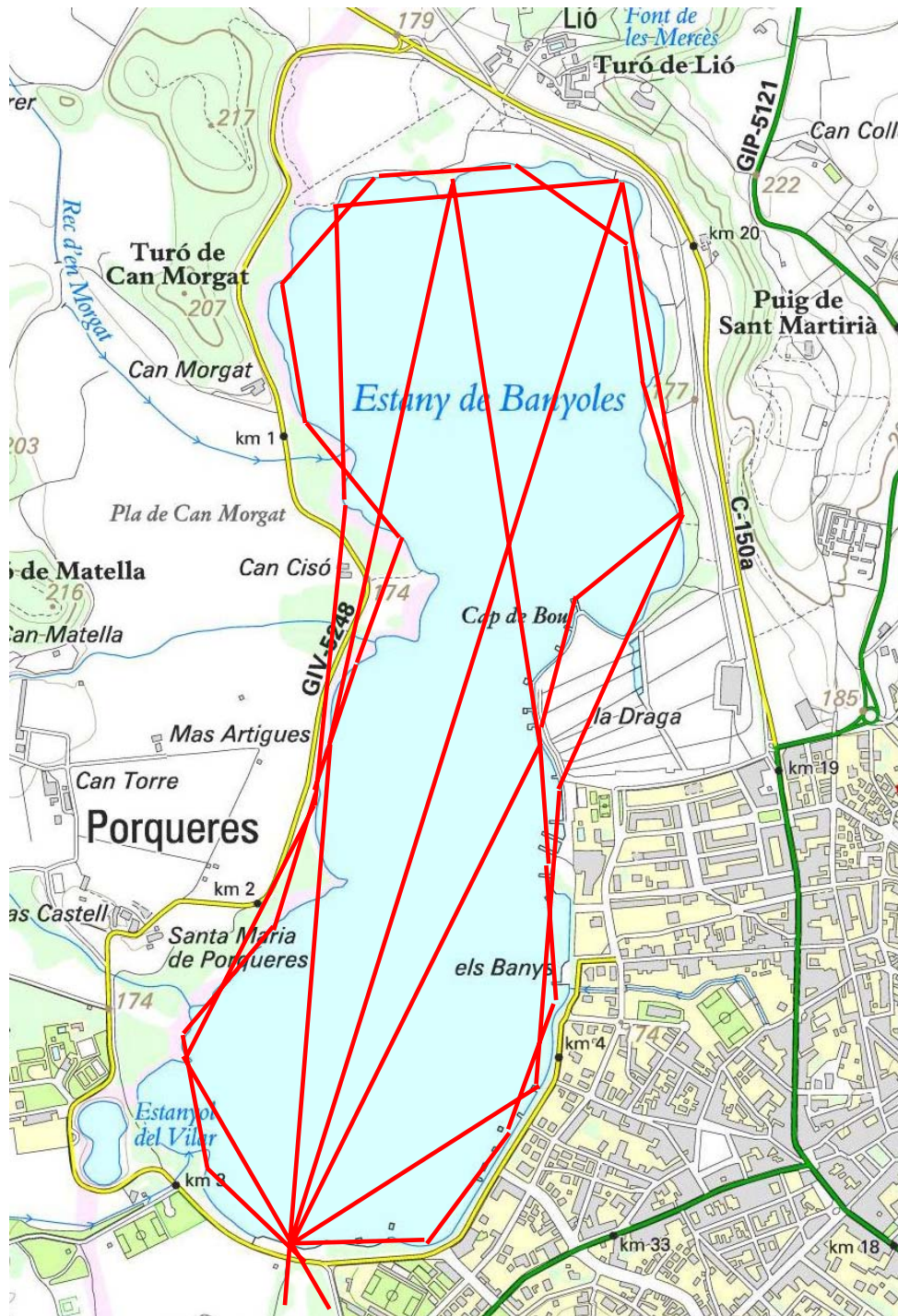
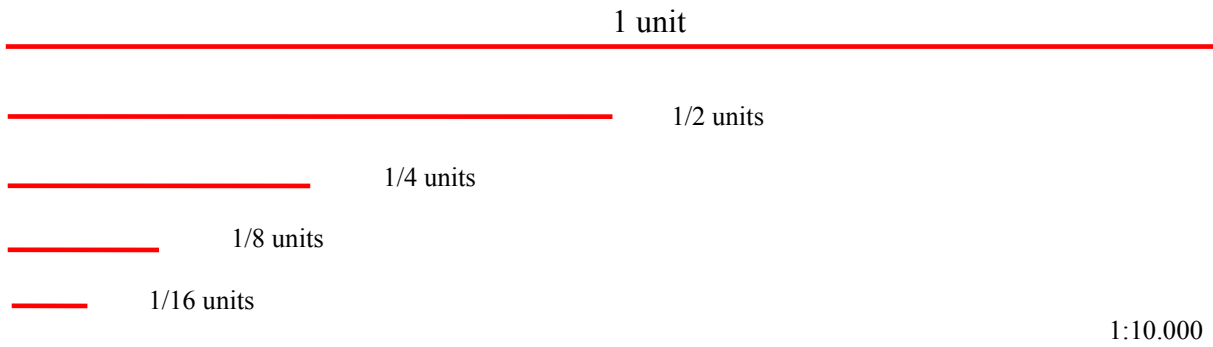
1 unit

- Compasses would be useful too.



Procedure: The Ruler method.

- Measure the length of the coastline using a collection of rulers whose lengths get shorter and shorter.
- Rewrite the data as a power of 10.
- **To minimize error, average the results from your group.**
- Plot the exponent data “length of ruler versus number of rulers needed”.
- You get a set of points that is almost linear.



1. After measuring the coastline from the map below, complete the table:

	Ruler length	Mean of Num of rulers
Stage 0	1	1
Stage 1	$\frac{1}{2} = 0.5$	3.8
Stage 2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$	8.7
Stage 3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8} = \mathbf{0.125}$	19.8
Stage 4	$\left(\frac{1}{2}\right)^4 = \frac{1}{16} = \mathbf{0.0625}$	40

The smaller the ruler, the more rulers we need to measure the coastline.

Notice that in the next questions we linearise data avoiding logarithms. Pupils can experiment with the log key in their calculators without knowing this specific concept.

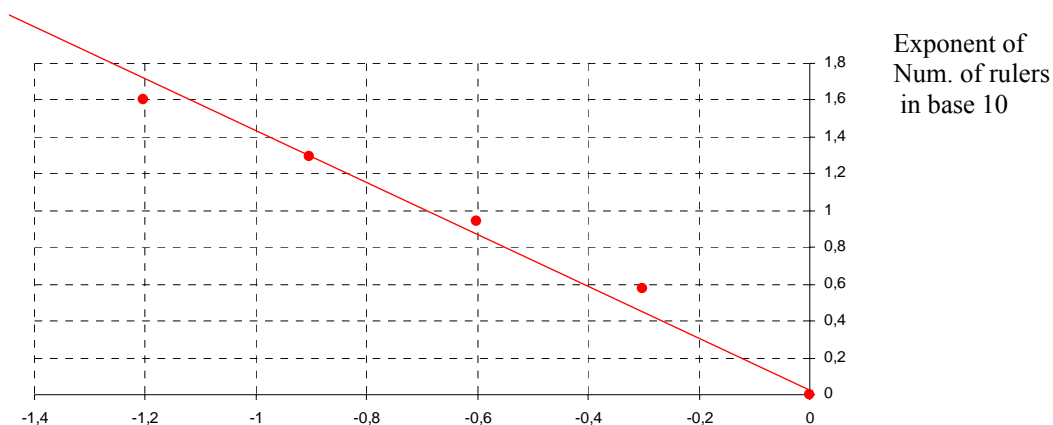
2. Experiment with the exponent key 10^x of your calculator to rewrite in base 10 the entries in the table above (correct to two decimal places):

	Ruler length	Num of rulers
Stage 0	10^0	10^0
Stage 1	$10^{-0.30}$	$10^{0.57}$
Stage 2	$10^{-0.60}$	$10^{0.93}$
Stage 3	$10^{-0.90}$	$10^{1.29}$
Stage 4	$10^{-1.20}$	$10^{1.60}$

3. Complete the table:

Exponent of ruler length in base 10	0	- 0.30	- 0.60	- 0.90	- 1.20
Exponent of Num of rulers in base 10	0	0.57	0.93	1.29	1.60

4. Sketch the graph of the exponents in the table above:



5. Calculate the ratio $\frac{\text{Exponent of Num. of rulers in base 10}}{\text{Exponent of ruler length in base 10}}$

$$\frac{0.57}{-0.30} \approx -1.88 \quad \frac{0.93}{-0.60} \approx -1.56 \quad \frac{1.29}{-0.90} \approx -1.43 \quad \frac{1.60}{-1.20} \approx -1.33$$

Actual plotted points do not all lie on a straight line, but they are very close to one.

You can take this opportunity to talk about direct proportion, slope of a straight line and line of best fit..

6. What is the fractal dimension of *Estany de Banyoles* coastline?

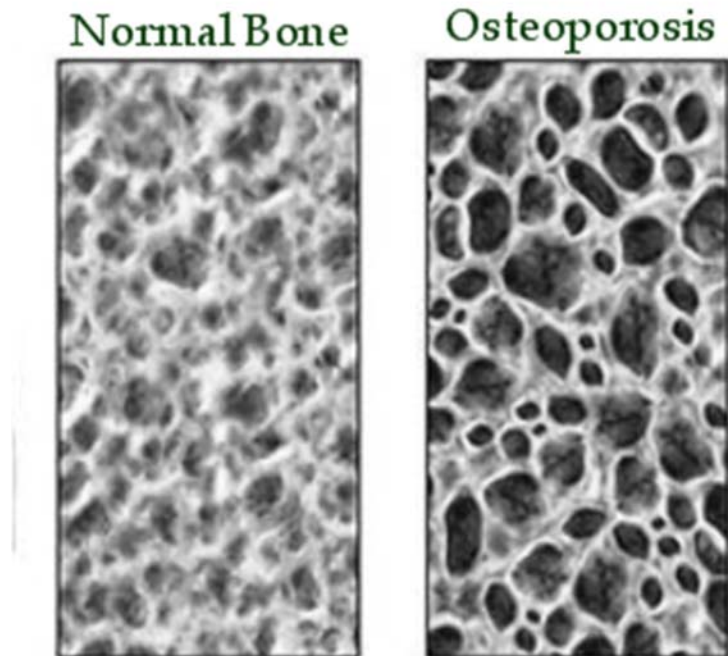
We do not get the exact value of the fractal dimension by this method. We only get an approximation. **Our solution is about 1.43.**

Pupils can get a better solution using a compass with a very sharp pencil, running two or three trials with the same ruler and averaging the results.

Next exercise is to test intuition for fractal dimension.

6-Bone cross-section

- Look at the digital images of a cross-section of bones



- Using the box counting method it is known that cross-section of normal bones have fractal dimension between 1.7 and 1.8.

- Osteoporosis is a condition of decreased bone mass. This leads to fragile bones which are at an increased risk for fractures.

- Choose the word Lower or Bigger to complete the sentence:

Lower fractal dimension is an index for the early diagnosis of osteoporosis.

Notice that the second image has many more holes than the first one.

Books

- Choate, J.; Devaney, R.L. & Foster, A. (1999). *Iteration: A Tool Kit of Dynamics Activities*. Key Curriculum Press.
- Choate, J.; Devaney, R.L. & Foster, A. (1999). *Fractals: A Tool Kit of Dynamics Activities*. Key Curriculum Press.
- Devaney, R. L. & Choate, J. (2000). *Chaos. A Tool Kit of Dynamics Activities*. Key Curriculum Press.

Web Sites

- <http://math.rice.edu/~lanius>
- <http://math.bu.edu/DYSYS>
- <http://www.xtec.cat/~dobrador>
- http://www.toomates.net/llestes/geometria/figures_geometriques/fractals/tetraedre_sierpinski_bastonets_gelat/tetraedre_sierpinski_bastonets_gelat.htm
- <http://www.mathsnet.net/puzzles/hanoi/index.html>
- <http://en.wikipedia.org/>
- http://en.wikipedia.org/wiki/Tower_of_Hanoi
- <http://www.lhs.berkeley.edu/Java/Tower/tower.html>
- http://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension
- <http://classes.yale.edu/Fractals/Panorama/welcome.html>
- <http://scidiv.bcc.ctc.edu/Math/snowflake.html>
- <http://www.42explore.com/fractal.htm>
- <http://www.shodor.org/interactivate/lessons/InfinityRecursion/>
- <http://www.fpolar.org.ve/matematica3/fasciculo26.pdf>
- http://www.math.fau.edu/Teacher/CATEs_PDF/3%20sierpinski/sierpinski%20unit%204.pdf
- <http://hypertextbook.com>
- <http://hypertextbook.com/chaos/33.shtml>

THE GEOMETRY OF NATURE: FRACTALS

- <http://hypertextbook.com/facts/2002/leaves.shtml>
- <http://hypertextbook.com/facts/2002/broccoli.shtml>
- <http://www1.cs.columbia.edu/~coms6998/Notes/lecture8.pdf>
- http://www.viamedica.pl/gazety/gazetax1ang/darmowy_pdf.phtml?indeks=24&indeks_art=336
- <http://www.3dfractals.com/docs/3DFractals.pdf>
- <http://www.jcu.edu/math/Vignettes/lsystems.htm>
- <http://www.science-house.org/student/bw/chaos/index.html>
- <http://www.edhelper.com/math.htm>
- http://www.cs.mu.oz.au/385/lectures/lecture3_DynSys3.pdf
- <http://positivodigital.blogspot.com/2008/08/descubren-la-causa-de-la-osteoporosis.html>
- <http://www.jimloy.com/fractals/koch.htm>